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The Lax–Milgram Theorem.
A detailed proof to be formalized in Coq

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Abstract: To obtain the highest confidence on the correction of numerical simulation programs implementing the finite element method, one has to formalize the mathematical notions and results that allow to establish the soundness of the method. The Lax–Milgram theorem may be seen as one of those theoretical cornerstones: under some completeness and coercivity assumptions, it states existence and uniqueness of the solution to the weak formulation of some boundary value problems. The purpose of this document is to provide the formal proof community with a very detailed pen-and-paper proof of the Lax–Milgram theorem.

Key-words: Lax–Milgram theorem, finite element method, detailed mathematical proof, formal proof in real analysis

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Le théorème de Lax–Milgram.
Une preuve détaillée en vue d'une formalisation en Coq

Résumé : Pour obtenir la plus grande confiance en la correction de programmes de simulation numérique implémentant la méthode des éléments finis, il faut formaliser les notions et résultats mathématiques qui permettent d'établir la justesse de la méthode. Le théorème de Lax–Milgram peut être vu comme l'un de ces fondements théoriques : sous des hypothèses de complétude et de coercivité, il énonce l'existence et l'unicité de la solution de certains problèmes aux limites posés sous forme faible. L'objectif de ce document est de fournir à la communauté preuve formelle une preuve papier très détaillée du théorème de Lax–Milgram.

Mots-clés : théorème de Lax–Milgram, méthode des éléments finis, preuve mathématique détaillée, preuve formelle en analyse réelle

Contents

1 Introduction

As stated and demonstrated in [4], formal proof tools are now mature to address the verification of scientific computing programs. One of the most thrilling aspects of the approach is that the round-off error due to the use of IEEE-754 floating-point arithmetic can be fully taken into account. One of the most important issue in terms of manpower is that all the mathematical notions and results that allow to establish the soundness of the implemented algorithm must be formalized.

The long term purpose of this study is to formally prove programs using the Finite Element Method. The Finite Element Method is now widely used to solve partial differential equations, and its success is partly due to its well established mathematical foundation, e.g. see [7, 14, 8, 17]. It seems important now to verify the scientific computing programs based on the Finite Element Method, in order to certify their results. The present report is a first contribution toward this ultimate goal.

The Lax–Milgram theorem is one of the key ingredients used to build the Finite Element Method. It is a way to establish existence and uniqueness of the solution to the weak formulation and its discrete approximation; it is valid for coercive linear operators set on Hilbert spaces (i.e. complete inner product spaces over the field of real or complex numbers). A corollary known as the Céa's lemma provides a quantification of the error between the computed approximation and the unknown solution. In particular, the Lax–Milgram theorem is sufficient to prove existence and uniqueness of the solution to the (weak formulation of the) standard Poisson problem defined as follows. Knowing a function f defined over a regular and bounded domain Ω of $\mathbb{R}^d$ with $d = 1$, 2, or 3, with its boundary denoted by $\partial\Omega$,

$$\text{find } u \text{ such that: } \left\{ \begin{array}{rcll} -\Delta u & = & f & \text{in } \Omega, \\ u & = & 0 & \text{on } \partial\Omega, \end{array} \right. \tag{1}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplace operator. Equation (1) is the strong formulation of Laplace problem, its weak formulation and the link with the Lax–Milgram theorem is given in Conclusion, perspectives, see Section 5. We do not intend to limit ourselves to this particular problem, but we stress that our work covers this standard problem that is the basis for the study of many other physical problems.

Other mathematical tools can be used to establish existence and uniqueness of the solution to weak problems. For instance, the Banach–Nečas–Babuška theorem for Banach spaces (i.e. complete normed real or complex vector spaces), from which one can deduce the Lax–Milgram theorem, e.g. see [8], or the theory for mixed and saddle-point problems, that is used for instance for some fluid problems, e.g. see [6, 9]. However, our choice is mainly guided by our limited manpower and by the intuitionistic logic of the interactive theorem prover we intend to use: we try to select an elementary and constructive path of proof. This advocates to work in a first step with Hilbert spaces rather than Banach spaces, and to try to avoid the use of Hahn–Banach theorem whose proof is based on Zorn's lemma (an equivalent of the axiom of choice in Zermelo–Fraenkel set theory).

Some other steps will be necessary for the formalization of the Finite Element Method: the measure theory is required to formalize Sobolev spaces such as $L^2(\Omega)$, $H^1(\Omega)$ and $H^1_0(\Omega)$ on some reasonable domain Ω, and establish that they are Hilbert spaces on which the Lax–Milgram theorem applies; as well as the notion of distribution to set up the correct framework to deal with weak formulations; and finally chapters of the interpolation and approximation theories to define the discrete finite element approximation spaces.

The purpose of this document is to provide the "formal proof" community with a very detailed pen-and-paper proof of the Lax–Milgram theorem. The most basic notions and results such as ordered field properties of $\mathbb{R}$ and properties of elementary functions over $\mathbb{R}$ are supposed known and are not detailed further. One of the key issues is to select in the literature the proof involving the simplest notions, and in particular not to justify the result by applying a more general statement. Once a detailed proof of the Lax–Milgram theorem is written, the next step is to formalize all notions and results in a formal proof tool such as Coq[1]. At this point, which is not the subject of the present paper, it will be necessary to take care of the specificities of the classical logic commonly used in mathematics: in particular, determine where there is need for the law of excluded middle, and discuss decidability issues.

The paper is organized as follows. Different ways to prove variants of the Lax–Milgram theorem collected from the literature are first reviewed in Section 2. The chosen proof path is then sketched in Section 3, and fully detailed in Section 4. Finally, lists of statements and direct dependencies are gathered in the appendix.

2 State of the art

We review some works of a few authors, mainly from the French school, that provide some details about statements similar to the Lax–Milgram theorem.

As usual, proofs provided in the literature are not comprehensive, and we have to cover a series of books to collect all the details necessary for a formalization in a formal proof tool such as Coq. Usually, Lecture Notes in undergraduate mathematics are very helpful and we selected [10, 11, 12] among many other possible choices.

2.1 Brézis

In [5], the Lax–Milgram theorem is stated as Corollary V.8 (p. 84). Its proof is obtained from Theorem V.6 (Stampacchia, p. 83) and by means similar to the ones used in the proof of Corollary V.4 (p. 80) for the characterization of the projection onto a closed subspace.

The proof of the Stampacchia theorem for a bilinear form on a closed convex set has four main arguments: the Riesz–Fréchet representation theorem (Theorem V.5 p. 81), the characterization of the projection onto a closed convex set (Theorem V.2 p. 79), the fixed point theorem on a complete metric space (Theorem V.7 p. 83), and the continuity of the projection onto a closed convex set (Proposition V.3 p. 80).

The proof of the fixed point theorem uses the notions of distance, completeness, and sequential continuity (e.g. see [11, Theorem 4.102 p. 115]).

The proof of the Riesz–Fréchet representation theorem and the existence of the projection onto a closed convex set share the possibility to use the notion of reflexive space through Proposition V.1 (p. 78) and Theorem III.29 (Milman–Pettis, p. 51). The latter states that uniformly convex Banach spaces are reflexive (i.e. isomorphic to their topological double dual), and its proof uses the notions of weak and weak-$\star$ topologies. In this case, the existence of the projection onto a closed convex set also needs the notions of compactness and lower semi-continuity through Corollary III.20 (p. 46), and the call to Hahn–Banach theorem which depends on Zorn's lemma or the axiom of choice through Theorem III.7 and Corollary III.8 (p. 38).

More elementary proofs are also presented in [5]. The Riesz–Fréchet theorem only needs the closed kernel lemma (for continuous linear maps) and the already cited Corollary V.4. The existence of the projection onto a closed convex set can be obtained through elementary and geometrical arguments. Then, the uniqueness and the characterization of the projection onto a closed convex set derives from the parallelogram identity and Cauchy–Schwarz inequality.

Complements about the projection onto a closed convex set subset can be found in [12, Lemmas 14.30 and 14.32 pp. 225–228]. See also [16, p. 90] and [8, Theorem A.28 p. 467] for proofs of the Riesz–Fréchet representation theorem.

2.2 Ciarlet

In [7], the Lax–Milgram theorem is stated as Theorem 1.1.3 (pp. 8–10). The structure of the proof is similar to the one proposed in [5] for Stampacchia theorem, but simplified to the case of a

[1]http://coq.inria.fr/

subspace instead of a closed convex subset (e.g. see [12, Theorems 14.27 and 14.29 pp. 224–225]).

2.3 Ern–Guermond

In [8], the Lax–Milgram theorem is stated as Lemma 2.2 (p. 83). The first proof is obtained as a consequence of the more general Banach–Nečas–Babuška theorem set on a Banach space (Theorem 2.6 p. 85). The proof is spread out in Section A.2 through Theorem A.43 (p. 472) for the characterization of bijective Banach operators, Lemma A.39 (p. 470) which is a consequence of the closed range theorem (Theorem A.34 p. 468, see also [16, pp. 205–208] and [5, p. 28]) and of the open mapping theorem (Theorem A.35 p. 469).

A simpler alternative proof without the use of the Banach–Nečas–Babuška theorem is proposed in Exercise 2.11 (p. 107) through the closed range theorem and a density argument. For the latter, Corollary A.18 (p. 466) is a consequence of the Hahn–Banach theorem (Theorem A.16 p. 465, see also [15, Theorem 5.19] and [5, p. 7]).

2.4 Quarteroni–Valli

In [14], the Lax–Milgram theorem is stated as Theorem 5.1.1 (p. 133). A variant, also known as Babuška–Lax–Milgram theorem, is stated for a bilinear form defined over two different Hilbert spaces (Theorem 5.1.2 p. 135). Their proofs are similar: they both use the Riesz–Fréchet representation theorem and the closed range theorem. Note that when the bilinear form is symmetric, the Riesz–Fréchet representation theorem and a minimization argument are sufficient to build the proof (Remark 5.1.1 p. 134).

3 Statement and sketch of the proof

Let H be a real Hilbert space. Let $(\cdot,\cdot)_H$ be its inner product, and $\|\cdot\|_H$ the associated norm. Let H' be its topological dual (i.e. the space of continuous linear forms on H). Let a be a bilinear form on H, and let $f \in H'$ be a continuous linear form on H. Let H_h be a closed vector subspace of H (in practice, H_h is finite dimensional). The Lax–Milgram theorem states existence and uniqueness of the solution to the following general problems:

$$\text{find } u \in H \text{ such that:} \qquad \forall v \in H, \quad a(u,v) = f(v); \tag{2}$$

$$\text{find } u_h \in H_h \text{ such that:} \qquad \forall v_h \in H_h, \quad a(u_h,v_h) = f(v_h). \tag{3}$$

The main statement is the following:

Lax–Milgram theorem. *Assume that a is bounded and coercive with constant $\alpha > 0$. Then, there exists a unique $u \in H$ solution to Problem (2). Moreover, $\|u\|_H \leq \frac{1}{\alpha}\|f\|_{H'}$.*

The ingredients for the proof were mainly collected from [5] and [7]. The key arguments of the chosen proof path are (the hierarchy is sketched in Figure 1):

- the representation lemma for bounded bilinear forms;
- the Riesz–Fréchet representation theorem;
- the orthogonal projection theorem for a complete subspace;
- the fixed point theorem for a contraction on a complete metric space.

Note that the same type of arguments can be used to prove the more general Stampacchia theorem. We give now more hints about the structure of the main steps of the proof.

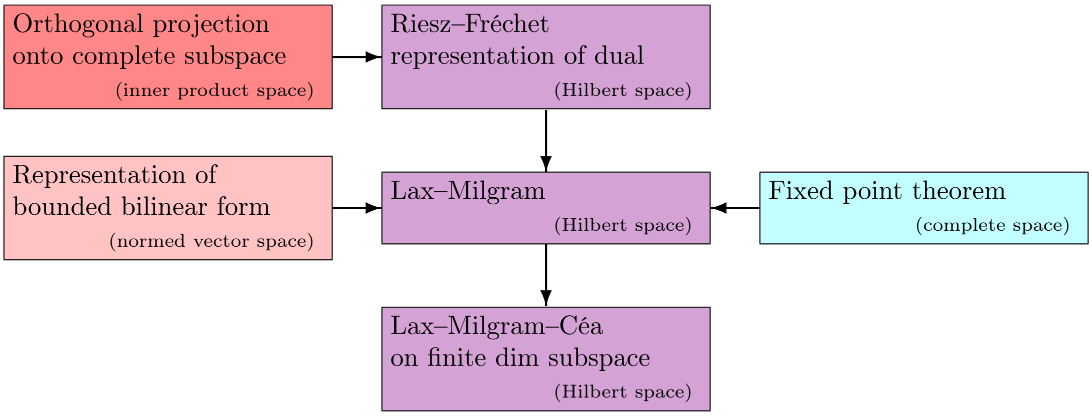

Figure 1: Hierarchy of results for a proof of the Lax–Milgram theorem.

3.1 Sketch of the proof of the Lax–Milgram–Céa theorem

Lax–Milgram–Céa theorem. *Assume that a is bounded with continuity constant $C \geq 0$ and coercive with constant $\alpha > 0$. Then, there exist a unique $u \in H$ solution to Problem (2), and a unique $u_h \in H_h$ solution to Problem (3). Moreover, $\|u\|_H \leq \frac{1}{\alpha}\|f\|_{H'}$ and for all $v_h \in H_h$, $\|u - u_h\|_H \leq \frac{C}{\alpha}\|u - v_h\|_H$.*

The proof of the Lax–Milgram–Céa theorem goes as follows (this proof uses the notion of finite dimensional subspace):

- a finite dimensional subspace is closed;
- a closed subspace of a Hilbert space is a Hilbert space;
- thus, the Lax–Milgram theorem applies to H_h;
- finally, Céa's error estimation is obtained from the Galerkin orthogonality property, and boundedness and coercivity of the bilinear form a.

3.2 Sketch of the proof of the Lax–Milgram theorem

Lax–Milgram theorem. *Assume that a is bounded and coercive with constant $\alpha > 0$. Then, there exists a unique $u \in H$ solution to Problem (2). Moreover, $\|u\|_H \leq \frac{1}{\alpha}\|f\|_{H'}$.*

The proof of the Lax–Milgram theorem goes as follows (this proof uses the notions of Lipschitz continuity, normed vector space, bounded and coercive bilinear form, inner product, orthogonal complement, and Hilbert space):

- Problem (2) is first shown to be equivalent to a fixed point problem for some contraction g on the complete metric space H:
  - the representation lemma for bounded bilinear forms exhibits a continuous linear form $A(u)$ such that $a(u, v) = (A(u))(v)$,
  - then, the Riesz–Fréchet representation theorem exhibits representatives $\tau(A(u))$ and $\tau(f)$ for both continuous linear forms, and Problem (2) is shown to be equivalent to the linear problem $\tau(A(u)) = \tau(f)$,
  - the affine function g is defined over H by $g(v) = v - \rho\tau(A(v)) + \rho\tau(f)$ for some small enough number ρ, then, Problem (2) is shown to be equivalent to find a fixed point of g,
  - finally, g is shown to be a contraction;

- thus existence and uniqueness of the solution u are obtained from the fixed point theorem applied to g;
- finally, the estimation of the nonzero solution u is obtained from the coercivity of the bilinear form a and the continuity of the linear form f.

3.3 Sketch of the proof of the representation lemma for bounded bilinear forms

Representation lemma for bounded bilinear forms. *Let $(E, \|\cdot\|_E)$ be a normed vector space. Let φ be a bilinear form on E. Assume that φ is bounded. Then, there exists a unique continuous linear map A from E to E' such that for all $u, v \in E$, $\varphi(u, v) = (A(u))(v)$. Moreover, for all $C \geq 0$ continuity constant of φ, we have $|||A|||_{E',E} \leq C$.*

The proof of the representation lemma for bounded bilinear forms goes as follows (this proof uses the notions of normed vector space, continuous linear map, topological dual, dual norm, and bounded bilinear form):

- existence of the representative A is obtained by construction:
  - for each u, the function $A_u = (v \mapsto \varphi(u, v))$ is shown to be a continuous linear form with $\|A_u\|_{E'} \leq C\ \|u\|_E$,
  - then, the function $A = (u \mapsto A_u)$ is shown to be a continuous linear map from E to E' with $|||A|||_{E',E} \leq C$;
- uniqueness of the representative A follows from the fact that continuous linear maps between two normed vector spaces form a normed vector space.

3.4 Sketch of the proof of the Riesz–Fréchet theorem

Riesz–Fréchet theorem. *Let $\varphi \in H'$ be a continuous linear form on H. Then, there exists a unique vector $u \in H$ such that for all $v \in H$, $\varphi(v) = (u, v)_H$. Moreover, the mapping $\tau = (\varphi \mapsto u)$ is a continuous isometric isomorphism from H' onto H.*

The proof of the Riesz–Fréchet representation theorem goes as follows (this proof uses the notions of kernel of a linear map, normed vector space, operator norm, continuous linear map, topological dual, dual norm, inner product space, orthogonal projection onto a complete subspace, orthogonal complement, and Hilbert space):

- uniqueness of the representative u_φ follows from the definiteness of the inner product;
- existence of the representative u_φ is obtained by construction for a nonzero φ:
  - consider the orthogonal projection onto $F = \ker(\varphi)$, which is closed, hence a complete subspace,
  - a unit vector ξ_0 in $F^\perp$ such that $\varphi(\xi_0) \neq 0$ is built from some u_0 picked in the complement of F, and using the theorem on the direct sum of a complete subspace and its orthogonal complement,
  - the candidate $u = \varphi(\xi_0)\xi_0 \in F^\perp$ is then shown to satisfy $(u, v)_H = \varphi(v)$;
- the mapping $\tau = (\varphi \mapsto u_\varphi)$ goes from the topological dual H' to the Hilbert space H, its linearity follows the bilinearity of the inner product and of the application of linear maps;
- injectivity of τ is straightforward, and surjectivity comes from Cauchy–Schwarz inequality;
- the isometric property of τ follows from the definition of the dual norm, and again from Cauchy–Schwarz inequality;
- continuity of τ follows from the isometric property.

3.5 Sketch of the proof of the orthogonal projection theorem for a complete subspace

Orthogonal projection theorem for a complete subspace. *Let $(G, (\cdot,\cdot)_G)$ be a real inner product space. Let F be a complete subspace of G. Then, for all $u \in G$, there exists a unique $v \in F$ such that $\|u - v\|_G = \min_{w \in F} \|u - w\|_G$.*

The proof of the orthogonal projection theorem for a complete subspace goes as follows (this proof uses the notions of infimum, completeness, inner product space, and convexity):

- the result is first shown for a nonempty complete convex subset K:
  - existence of the projection of $u \in G$ onto K is built as the limit of a sequence:
    - existence of a sequence $(w_n)_{n \in \mathbb{N}}$ in K and a nonnegative number δ such that $\|u - w_n\|_G < \delta + \frac{1}{n+1}$ is first obtained from the fact that the function $(w \mapsto \|u - w\|_G)$ is bounded from below (by 0),
    - this sequence is shown to be a Cauchy sequence using the parallelogram identity and the definition of convexity,
    - hence, it is convergent in the complete subset K,
    - continuity of the norm ensures that the limit of the sequence realizes the minimum of the distance;
  - uniqueness of the projection follows again the parallelogram identity and the definition of convexity;
- a complete subspace is also a nonempty complete convex subset.

3.6 Sketch of the proof of the fixed point theorem

Fixed point theorem. *Let (X, d) be a complete metric space. Let $f : X \to X$ be a contraction. Then, there exists a unique fixed point $a \in X$ such that $f(a) = a$. Moreover, all iterated function sequences associated with f are convergent with limit a.*

The proof of the fixed point theorem goes as follows (this proof uses the notions of distance, completeness and Lipschitz continuity).

- uniqueness of the fixed point is obtained from the properties of the distance;
- existence of the fixed point is built from the sequence of iterates of the contraction:
  - when nonstationary, the sequence is first proved to be a Cauchy sequence using the iterated triangle inequality and the formula for the sum of the first terms of a geometric series,
  - the sequence is then convergent in a complete metric space,
  - the limit of the sequence is finally proved to be a fixed point of the contraction from properties of the contraction and of the distance.

4 Detailed proof

The Lax–Milgram theorem is stated on a Hilbert space. The notion of Hilbert space is built from a series of notions of spaces, see Figure 2 for a sketch of the hierarchy. Thus, a large part of the present section collects standard definitions and results from linear and bilinear algebra. One of the main steps is the construction of the complete normed space of continuous linear maps, used in particular to obtain the notion of topological dual. Then, another step is the construction of bilinear forms, and their representation as linear maps with values in the topological dual. The

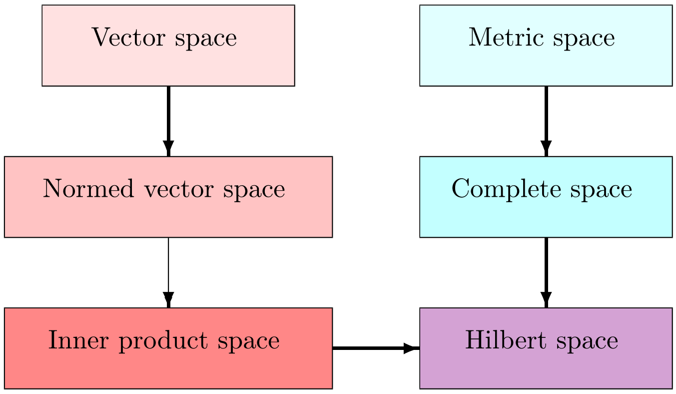

Figure 2: Hierarchy of notions used to build a Hilbert space. Thick arrows indicate inheritance: the target notion is built upon the source one, whereas the lower left thin arrow indicates that inner product spaces are only shown to be normed vector spaces.

results on finite dimensional vector spaces are avoided as much as possible, as well as the results on Banach spaces.

A set of basic results from topology in metric spaces are necessary to formulate the fixed point theorem, and to link completeness and closedness, which is useful in particular to characterize finite dimensional spaces as complete.

The last statement of this document is dedicated to the finite dimensional case. To prove Theorem 203 (Lax–Milgram–Céa), we use the closedness of finite dimensional subspaces, which is a direct consequence of the closedness of the sum of a closed subspace and a linear span. Such a result is of course valid in any normed vector space, but the general proof is based on the equivalence of norms in a finite dimensional space, and the latter needs more advanced results on continuity involving compactness. To avoid that, we propose a much simpler proof that is only valid in inner product spaces; which is fine here since we apply it on a Hilbert space.

Statements are displayed inside colored boxes. Their nature can be identified at a glance by using the following color code:

light gray is for remarks, light green for definitions,
light blue for lemmas, and light red for theorems.

Moreover, inside the bodies of proof for lemmas and theorems, the most basic results are supposed to be known and are not detailed further; they are displayed in **bold red**. This includes:

- properties from propositional calculus;
- basic notions and results from set theory such as the complement of a subset, the composition of functions, injective and surjective functions;
- basic results from group theory;
- ordered field properties of $\mathbb{R}$, ordered set properties of $\overline{\mathbb{R}}$;
- basic properties of the complete valued fields $\mathbb{R}$ and $\mathbb{C}$;
- definition and properties of basic functions over $\mathbb{R}$ such the square, square root, and exponential functions, and the discriminant of a quadratic polynomial;
- basic properties of geometric series (sum of the first terms).

This section is organized as follows. Some facts about infima and suprema are first collected in Section 4.1, they are useful to define the operator norm for continuous linear maps, and orthogonal projections in inner product spaces. Then, Section 4.2 is devoted to complements on metric spaces, it concludes with the fixed point theorem. Section 4.3 is for the general notion of vector spaces. Normed vector spaces are introduced in Section 4.4, with the continuous linear map equivalency theorem. Then, we add an inner product in Section 4.5 to define inner product spaces and state a series of orthogonal projection theorems. Finally, Section 4.6 is dedicated to Hilbert spaces with the Riesz–Fréchet representation theorem, and variants of the Lax–Milgram theorem.

4.1 Supremum, infimum

Remark 1. From the completeness of the set of real numbers, every nonempty subset of $\mathbb{R}$ that is bounded from above has a least upper bound in $\mathbb{R}$. On the affinely extended real number system $\overline{\mathbb{R}}$, every nonempty subset has a least upper bound that may be $+\infty$ (and a greatest lower bound that may be $-\infty$). Thus, we have the following extended notions of supremum and infimum for a numerical function over a set.

Definition 2 ***(supremum).*** Let X be a set. Let $f : X \to \mathbb{R}$ be a function. The extended number L is the *supremum of* f *over* X, and is denoted $L = \sup(f(X))$, iff it is the least upper bound of $f(X) = \{f(x) \,|\, x \in X\} \subset \mathbb{R}$:

$$\forall x \in X, \qquad f(x) \le L; \tag{4}$$

$$\forall M \in \overline{\mathbb{R}}, \qquad (\forall x \in X, \quad f(x) \le M) \quad \Longrightarrow \quad L \le M. \tag{5}$$

Lemma 3 ***(finite supremum).*** *Let X be a set. Let $f : X \to \mathbb{R}$ be a function. Assume that there exists a finite upper bound for $f(X)$, i.e. there exists $M \in \mathbb{R}$ such that, for all $x \in X$, $f(x) \le M$. Then, the supremum is finite and $L = \sup(f(X))$ iff (4) and*

$$\forall \varepsilon > 0, \exists x_\varepsilon \in X, \quad L - \varepsilon < f(x_\varepsilon). \tag{6}$$

Proof. From hypothesis, and completeness of $\mathbb{R}$, $\sup(f(X))$ is finite. Let L be a number. Assume that L is an upper bound of $f(X)$, i.e. (4).

(5) implies (6). Assume that (5) holds. Let $\varepsilon > 0$. Suppose that for all $x \in X$, $f(x) \le L - \varepsilon$. Then, $L - \varepsilon$ is an upper bound for $f(X)$. Thus, from hypothesis, we have $L \le L - \varepsilon$, and from **ordered field properties of** $\mathbb{R}$, $\varepsilon \le 0$, which is impossible. Hence, there exists $x \in X$, such that $L - \varepsilon < f(x)$.

(6) implies (5). Conversely, assume now that (6) holds. Let M be an upper bound, i.e. for all $x \in X$, $f(x) \le M$. Suppose that $M < L$. Let $\varepsilon = \frac{L-M}{2} > 0$. Then, from hypotheses, and **ordered field properties of** $\mathbb{R}$, there exists $x_\varepsilon \in X$ such that $f(x_\varepsilon) > L - \varepsilon = \frac{L+M}{2} > M$, which is impossible. Hence, we have $L \le M$.

Therefore, (4) implies the equivalence between (5) and (6). □

Lemma 4 ***(discrete lower accumulation).*** *Let X be a set. Let $f : X \to \mathbb{R}$ be a function. Let L be a number. Then, (6) iff*

$$\forall n \in \mathbb{N}, \exists x_n \in X, \quad L - \frac{1}{n+1} < f(x_n). \tag{7}$$

Proof. **(6) implies (7).** Assume that (6) holds. Let $n \in \mathbb{N}$. Let $\varepsilon = \frac{1}{n+1} > 0$. Then, from hypothesis, there exists $x_n = x_\varepsilon \in X$ such that

$$L - \frac{1}{n+1} = L - \varepsilon < f(x_\varepsilon) = f(x_n).$$

(7) implies (6). Conversely, assume now that (7) holds. Let $\varepsilon > 0$. From the **Archimedean property of $\mathbb{R}$**, there exists $n \in \mathbb{N}$ such that $n > \frac{1}{\varepsilon} - 1$ (e.g. $n = \lfloor \frac{1}{\varepsilon} \rfloor$). Then, from **ordered field properties of $\mathbb{R}$**, and hypothesis, we have $\varepsilon > \frac{1}{n+1}$, and there exists $x_\varepsilon = x_n \in X$ such that

$$L - \varepsilon < L - \frac{1}{n+1} < f(x_n) = f(x_\varepsilon).$$

$\square$

Lemma 5 ***(supremum is positive scalar multiplicative).*** *Let X be a set. let $f : X \to \mathbb{R}$ be a function. Let $\lambda \geq 0$ be a nonnegative number. Then,* $\sup((\lambda f)(X)) = \lambda \sup(f(X))$ *(**with the convention that 0 times $+\infty$ is 0**).*

Proof. Let $L = \sup(f(X))$ and $M = \sup((\lambda f)(X))$ be extended numbers of $\overline{\mathbb{R}}$. Let $x \in X$.

From Definition 2 (*supremum*, L is an upper bound for $f(X)$), and **ordered set properties of $\overline{\mathbb{R}}$**, we have

$$(\lambda f)(x) = \lambda f(x) \leq \lambda L.$$

Thus, λL is an upper bound of $(\lambda f)(X)$. Hence, from Definition 2 (*supremum*, M is the least upper bound of $(\lambda f)(X)$), we have $M \leq \lambda L$.

Case $\boldsymbol{\lambda = 0}$. Then, λf is the zero function (for all $x \in X$, $(\lambda f)(x) = \lambda f(x) = 0$). Thus, $(\lambda f)(X) = \{0\}$, and from Definition 2 (*supremum*, 0 is the least upper bound of $\{0\}$), $M = 0$. Hence, from **ordered set properties of $\overline{\mathbb{R}}$**, we have

$$\lambda L = 0\, L = 0 \leq 0 = M.$$

Case $\boldsymbol{\lambda \neq 0}$. Then, from hypothesis, $\lambda > 0$. From Definition 2 (*supremum*, M is an upper bound for $(\lambda f)(X)$), **field properties of $\mathbb{R}$**, and **ordered set properties of $\overline{\mathbb{R}}$**, we have

$$f(x) = \frac{1}{\lambda}\lambda f(x) = \frac{1}{\lambda}(\lambda f)(x) \leq \frac{M}{\lambda}.$$

Thus, $\frac{M}{\lambda}$ is an upper bound for $f(X)$. Hence, from Definition 2 (*supremum*, L is the least upper bound of $f(X)$), and **ordered set properties of $\overline{\mathbb{R}}$**, we have $L \leq \frac{M}{\lambda}$, or equivalently $\lambda L \leq M$.

Therefore, $M = \lambda L$. $\square$

Remark 6. As a consequence, $\sup((\lambda f)(X))$ is finite iff $\lambda\sup(f(X))$ is finite.

Definition 7 ***(maximum).*** Let X be a set. Let $f : X \to \mathbb{R}$ be a function. The supremum of f over X is called *maximum of f over X*, and it is denoted $\max(f(X))$, iff there exists $y \in X$ such that $f(y) = \sup(f(X))$.

Lemma 8 ***(finite maximum).*** *Let X be a set. Let $f : X \to \mathbb{R}$ be a function. Let $y \in X$. Then,*

$$f(y) = \max(f(X)) \iff \forall x \in X, \quad f(x) \leq f(y). \tag{8}$$

Proof. **"Left" implies "right".** Assume that y realizes the maximum of f over X. Let $x \in X$. Then, from hypothesis, Definition 7 (*maximum*), and Definition 2 (*supremum*, $f(y)$ is an upper bound of $f(X)$), we have $f(x) \leq f(y)$.

"Right" implies "left". Conversely, assume now that $f(y)$ is an upper bound of $f(X)$. Let $\varepsilon > 0$. Let $x_\varepsilon = y \in X$. Then, from **ordered field properties of $\mathbb{R}$**, we have $f(y) - \varepsilon < f(y) = f(x_\varepsilon)$. Hence, from Lemma 3 (*finite supremum*), and Definition 7 (*maximum*), we have $f(y) = \max(f(X))$. $\square$

Definition 9 *(infimum).* Let X be a set. Let $f : X \to \mathbb{R}$ be a function. The extended number l is the *infimum of f over X*, and is denoted $l = \inf(f(X))$, iff it is the greatest lower bound of $f(X) \subset \mathbb{R}$:

$$\forall x \in X, \qquad l \leq f(x); \tag{9}$$

$$\forall m \in \overline{\mathbb{R}}, \qquad (\forall x \in X, \quad m \leq f(x)) \quad \Longrightarrow \quad m \leq l. \tag{10}$$

Lemma 10 *(duality infimum-supremum).* *Let X be a set. Let $f : X \to \mathbb{R}$ be a function. Then,* $\inf(f(X)) = -\sup((-f)(X))$.

Proof. Let $L = \sup((-f)(X))$ be an extended number in $\overline{\mathbb{R}}$. Let $l = -L$.

Let $x \in X$. From Definition 2 (*supremum*, L is an upper bound of $(-f)(X)$), and **ordered set properties of $\overline{\mathbb{R}}$**, we have $l = -L \leq f(x)$. Hence, l is a lower bound of $f(X)$.

Let $m \in \overline{\mathbb{R}}$ be a lower bound of $f(X)$. Let $x \in X$. Then, from **ordered set properties of $\overline{\mathbb{R}}$**, we have $-f(x) \leq -m$. Thus, from Definition 2 (*supremum*, L is the least upper bound of $(-f)(X)$), and **ordered set properties of $\overline{\mathbb{R}}$**, we have $m \leq -L = l$. Hence, l is the greatest lower bound of $f(X)$.

Therefore, from Definition 9 (*infimum*), $l = \inf(f(X))$. □

Lemma 11 *(finite infimum).* *Let X be a set. Let $f : X \to \mathbb{R}$ be a function. Assume that there exists a finite lower bound for $f(X)$, i.e. there exists $m \in \mathbb{R}$ such that, for all $x \in X$, $m \leq f(x)$. Then, the infimum is finite and $l = \inf(f(X))$ iff* (9) *and*

$$\forall \varepsilon > 0, \exists x_\varepsilon \in X, \quad f(x_\varepsilon) < l + \varepsilon. \tag{11}$$

Proof. Let $x \in X$. Then, from hypothesis, and **ordered field properties of $\mathbb{R}$**, we have $-f(x) \leq -m$. Thus, from Lemma 3 (*finite supremum*), $\sup((-f)(X))$ is finite. Hence, from Lemma 10 (*duality infimum-supremum*), $\inf(f(X)) = -\sup((-f)(X))$ is finite too.

From Lemma 10 (*duality infimum-supremum*), Lemma 3 (*finite supremum*), and **ordered field properties of $\mathbb{R}$**, we have

$$\begin{aligned} l = \inf(f(X)) \quad &\Leftrightarrow \quad -l = \sup((-f)(X)) \\ &\Leftrightarrow \quad \left\{ \begin{array}{l} \forall x \in X, \quad -f(x) \leq -l \\ \forall \varepsilon > 0, \exists x_\varepsilon \in X, \quad -l - \varepsilon < -f(x_\varepsilon) \end{array} \right. \\ &\Leftrightarrow \quad \left\{ \begin{array}{l} \forall x \in X, \quad l \leq f(x) \\ \forall \varepsilon > 0, \exists x_\varepsilon \in X, \quad f(x_\varepsilon) < l + \varepsilon. \end{array} \right. \end{aligned}$$

□

Lemma 12 *(discrete upper accumulation).* *Let X be a set. Let $f : X \to \mathbb{R}$ be a function. Let l be a number. Then,*

$$\forall \varepsilon > 0, \exists x_\varepsilon \in X, \quad f(x_\varepsilon) < l + \varepsilon \quad \Longleftrightarrow \quad \forall n \in \mathbb{N}, \exists x_n \in X, \quad f(x_n) < l + \frac{1}{n+1}. \tag{12}$$

Proof. Direct consequence of Lemma 4 (*discrete lower accumulation*, for $-f$ and $L = -l$), and **ordered field properties of $\mathbb{R}$**. □

Lemma 13 *(finite infimum discrete).* *Let X be a set. Let $f : X \to \mathbb{R}$ be a function. Assume that there exists a finite lower bound for $f(X)$, i.e. there exists $m \in \mathbb{R}$ such that, for all $x \in X$, $m \leq f(x)$. Then, the infimum is finite and $l = \inf(f(X))$ iff* (9) *and*

$$\forall n \in \mathbb{N}, \exists x_n \in X, \quad f(x_n) < l + \frac{1}{n+1}. \tag{13}$$

Proof. Direct consequence of Lemma 11 (*finite infimum*), and Lemma 12 (*discrete upper accumulation*). □

Definition 14 *(minimum).* Let X be a set. Let $f : X \to \mathbb{R}$ be a function. The infimum of f over X is called *minimum of f over X* and it is denoted $\min(f(X))$, iff there exists $y \in X$ such that $f(y) = \inf(f(X))$.

Lemma 15 *(finite minimum).* *Let X be a set. Let $f : X \to \mathbb{R}$ be a function. Let $y \in X$. Then,*

$$f(y) = \min(f(X)) \iff \forall x \in X, \quad f(y) \leq f(x). \tag{14}$$

Proof. From Definition 14 (*minimum*), Lemma 10 (*duality infimum-supremum*), Lemma 8 (*finite maximum*), and **ordered field properties of $\mathbb{R}$**, we have

$$\begin{aligned} f(y) = \min(f(X)) &\Leftrightarrow -f(y) = \max((-f)(X)) \\ &\Leftrightarrow \forall x \in X, \quad -f(x) \leq -f(y) \\ &\Leftrightarrow \forall x \in X, \quad f(y) \leq f(x). \end{aligned}$$

□

4.2 Metric space

Definition 16 *(distance).* Let X be a nonempty set. An application $d : X \times X \to \mathbb{R}$ is a *distance over X* iff it is nonnegative, symmetric, it separates points, and it satisfies the triangle inequality:

$$\forall x, y \in X, \qquad d(x, y) \geq 0; \tag{15}$$

$$\forall x, y \in X, \qquad d(y, x) = d(x, y); \tag{16}$$

$$\forall x, y \in X, \qquad d(x, y) = 0 \iff x = y; \tag{17}$$

$$\forall x, y, z \in X, \qquad d(x, z) \leq d(x, y) + d(y, z). \tag{18}$$

Definition 17 *(metric space).* (X, d) is a *metric space* iff X is a nonempty set and d is a distance over X.

Lemma 18 *(iterated triangle inequality).* *Let (X, d) be a metric space. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points of X. Then,*

$$\forall n, p \in \mathbb{N}, \quad d(x_n, x_{n+p}) \leq \sum_{i=0}^{p-1} d(x_{n+i}, x_{n+i+1}). \tag{19}$$

Proof. Let $n \in \mathbb{N}$ be a natural number. For $p \in \mathbb{N}$, let $P(p)$ be the property

$$d(x_n, x_{n+p}) \leq \sum_{i=0}^{p-1} d(x_{n+i}, x_{n+i+1}).$$

Induction: $P(0)$. From Definition 16 (*distance*, d separates points), and **ordered field properties of $\mathbb{R}$**, $P(0)$ is obviously satisfied.

Induction: $P(p)$ implies $P(p+1)$. Let $p \in \mathbb{N}$. Assume that $P(p)$ holds. Then, from Definition 16 (*distance*, d satisfies triangle inequality), we have $d(x_n, x_{n+p+1}) \leq d(x_n, x_{n+p}) + d(x_{n+p}, x_{n+p+1})$. Hence, from hypothesis, and **ordered field properties of $\mathbb{R}$**, we have $P(p+1)$.

Therefore, by induction on $p \in \mathbb{N}$, we have, for all $p \in \mathbb{N}$, $P(p)$. □

4.2.1 Topology of balls

Definition 19 ***(closed ball).*** Let (X, d) be a metric space. Let $x \in X$ be a point. Let $r \geq 0$ be a nonnegative number. The *closed ball centered in* x *of radius* r, denoted $\mathcal{B}_d^c(x, r)$, is the subset of X defined by

$$\mathcal{B}_d^c(x, r) = \{y \in X \,|\, d(x, y) \leq r\}\,. \tag{20}$$

Definition 20 ***(sphere).*** Let (X, d) be a metric space. Let $x \in X$ be a point. Let $r \geq 0$ be a nonnegative number. The *sphere centered in* x *of radius* r, denoted $\mathcal{S}_d(x, r)$, is the subset of X defined by

$$\mathcal{S}_d(x, r) = \{y \in X \,|\, d(x, y) = r\}\,. \tag{21}$$

Definition 21 ***(open subset).*** Let (X, d) be a metric space. A subset Y of X is *open (for distance* d*)* iff

$$\forall x \in Y, \exists r > 0, \quad \mathcal{B}_d^c(x, r) \subset Y. \tag{22}$$

Definition 22 ***(closed subset).*** Let (X, d) be a metric space. A subset Y of X is *closed (for distance* d*)* iff $X \backslash Y$ is open for distance d.

Lemma 23 ***(equivalent definition of closed subset).*** *Let* (X, d) *be a metric space. A subset* Y *of* X *is* closed (for distance d) *iff*

$$\forall x \in X \backslash Y, \exists r > 0, \quad \mathcal{B}_d^c(x, r) \cap Y = \emptyset. \tag{23}$$

Proof. Direct consequence of Definition 22 (*closed subset*), Definition 21 (*open subset*), and **the definition of the complement from set theory**. □

Lemma 24 ***(singleton is closed).*** *Let* (X, d) *be a metric space. Let* $x \in X$ *be a point. Then* $\{x\}$ *is closed.*

Proof. Let $x' \in X$ be a point. Assume that $x' \neq x$. Then, from Definition 16 (*distance*, d separates points, contrapositive), and **ordered field properties of $\mathbb{R}$**, $\varepsilon = \frac{1}{2}\,d(x', x)$ is positive. Hence, $d(x', x) > \varepsilon$ and $\mathcal{B}_d^c(x', \varepsilon) \cap \{x\} = \emptyset$.

Therefore, from Lemma 23 (*equivalent definition of closed subset*), $\{x\}$ is closed. □

Definition 25 ***(closure).*** Let (X, d) be a metric space. Let Y be a subset of X. The *closure of* Y, denoted $\overline{Y}$, is the subset

$$\overline{Y} = \{x \in X \,|\, \forall \varepsilon > 0, \mathcal{B}_d^c(x, \varepsilon) \cap Y \neq \emptyset\}\,. \tag{24}$$

Definition 26 ***(convergent sequence).*** Let (X, d) be a metric space. Let $l \in X$. A sequence $(x_n)_{n \in \mathbb{N}}$ of X is *convergent with limit* l iff

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N}, \quad n \geq N \quad \Longrightarrow \quad d(x_n, l) \leq \varepsilon. \tag{25}$$

Lemma 27 ***(variant of point separation).*** *Let* (X, d) *be a metric space. Let* $x, x' \in X$ *such that for all* $\varepsilon > 0$*, we have* $d(x, x') \leq \varepsilon$*. Then,* $x = x'$*.*

Proof. Assume that $d(x, x') > 0$. Let $\varepsilon = \frac{d(x,x')}{2}$. Then, $0 < d(x, x') \leq \varepsilon = \frac{d(x,x')}{2}$. Hence, from **ordered field properties of $\mathbb{R}$ (with $d(x, x') > 0$)**, we have $0 < 1 \leq \frac{1}{2}$, which is wrong. Thus, from Definition 16 (*distance*, d is nonnegative), we have $d(x, x') = 0$.

Therefore, from Definition 16 (*distance*, d separates points), we have $x = x'$. □

Lemma 28 ***(limit is unique).*** *Let* (X, d) *be a metric space. Let* $(x_n)_{n \in \mathbb{N}}$ *be a convergent sequence of* X*. Then, the limit of the sequence is unique. The limit is denoted* $\lim_{n \to +\infty} x_n$*.*

Proof. Let $l, l' \in X$ be two limits of the sequence. Let $\varepsilon > 0$. Then, from **ordered field properties of $\mathbb{R}$**, and Definition 26 (*convergent sequence*, with $\frac{\varepsilon}{2} > 0$), let $N, N' \in \mathbb{N}$ such that, for all $n, n' \in \mathbb{N}$, $n \geq N$ and $n' \geq N'$ implies $d(x_n, l) \leq \frac{\varepsilon}{2}$ and $d(x_{n'}, l') \leq \frac{\varepsilon}{2}$. Let $M = \max(N, N')$. Let $p \in \mathbb{N}$. Assume that $p \geq M$. Then, from **the definition of the maximum**, we have $d(x_p, l) \leq \frac{\varepsilon}{2}$ and $d(x_p, l') \leq \frac{\varepsilon}{2}$. Hence, from Definition 16 (*distance*, d is nonnegative, satisfies triangle inequality, and is symmetric), and **ordered field properties of $\mathbb{R}$**, we have

$$0 \leq d(l, l') \leq d(l, x_p) + d(x_p, l') = d(x_p, l) + d(x_p, l') \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, from Lemma 27 (*variant of point separation*), we have $l = l'$. □

Lemma 29 *(closure is limit of sequences).* *Let (X, d) be a metric space. Let Y be a nonempty subset of X. Let $a \in X$ be a point. Then,*

$$a \in \overline{Y} \quad \Longleftrightarrow \quad \exists (a_n)_{n\in\mathbb{N}} \in Y^{\mathbb{N}}, \quad a = \lim_{n\to\infty} a_n. \tag{26}$$

Proof. **"Left" implies "right".** Assume that $a \in \overline{Y}$. Let a_0 be a point of Y. Let $n \in \mathbb{N}$. Assume that $n \geq 1$. Then, $\frac{1}{n} > 0$, and from Definition 25 (*closure*), let a_n be in the nonempty intersection $\mathcal{B}^c_d(a, \frac{1}{n}) \cap Y$. From Definition 19 (*closed ball*), and Definition 16 (*distance*, d is symmetric), we have $a_n \in Y$ and $d(a_n, a) \leq \frac{1}{n}$. Let $\varepsilon > 0$. Let $N = \left\lceil \frac{1}{\varepsilon} \right\rceil$. Let $n \in \mathbb{N}$. Assume that $n \geq N$. Then, from **ordered field properties of $\mathbb{R}$**, and **the definition of ceiling function**, we have

$$d(a_n, a) \leq \frac{1}{n} \leq \frac{1}{N} \leq \varepsilon.$$

Hence, from Definition 26 (*convergent sequence*), the sequence $(a_n)_{n\in\mathbb{N}}$ is convergent with limit a.

"Right" implies "left". Assume now that there exists a convergent sequence $(a_n)_{n\in\mathbb{N}}$ in Y with limit a. Let $\varepsilon > 0$. Then, from Definition 26 (*convergent sequence*), let $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $n \geq N$ implies $d(a_n, a) \leq \varepsilon$. Thus, a_N belongs to the ball $\mathcal{B}^c_d(a, \varepsilon)$. Hence, from Definition 25 (*closure*), a belongs to the closure $\overline{Y}$. □

Lemma 30 *(closed equals closure).* *Let (X, d) be a metric space. Let Y be a nonempty subset of X. Then,*

$$Y \text{ is closed} \quad \Longleftrightarrow \quad Y = \overline{Y}. \tag{27}$$

Proof. **"Left" implies "right".** Assume that Y is closed. Then, from Definition 22 (*closed subset*), $X \backslash Y$ is open. Let $a \in \overline{Y}$. Then, from Definition 25 (*closure*), for all $\varepsilon > 0$, we have $\mathcal{B}^c_d(a, \varepsilon) \cap Y \neq \emptyset$. Assume that $a \notin Y$. Then, from **the definition of the complement from set theory**, and Lemma 23 (*equivalent definition of closed subset*), there exists $\varepsilon > 0$ such that $\mathcal{B}^c_d(a, \varepsilon) \cap Y = \emptyset$. Which is impossible. Thus, a belongs to Y. Hence, $\overline{Y} \subset Y$. Moreover, from Definition 25 (*closure*), Y is obviously a subset of $\overline{Y}$. Therefore, $Y = \overline{Y}$.

"Right" implies "left". Assume now that $Y = \overline{Y}$. Let $x \in X \backslash Y$. From **the definition of the complement from set theory**, and hypothesis, x does not belong to $Y = \overline{Y}$. Thus, from Definition 25 (*closure*), there exists $\varepsilon > 0$ such that $\mathcal{B}^c_d(x, \varepsilon) \cap Y = \emptyset$. Hence, from Lemma 23 (*equivalent definition of closed subset*), Y is closed. □

Lemma 31 *(closed is limit of sequences).* *Let (X, d) be a metric space. Let Y be a nonempty subset of X. Then,*

$$Y \text{ is closed} \quad \Longleftrightarrow \quad (\forall (a_n)_{n\in\mathbb{N}} \in Y^{\mathbb{N}}, \forall a \in X, \quad a = \lim_{n\to\infty} a_n \quad \Longrightarrow \quad a \in Y). \tag{28}$$

Proof. Direct consequence of Lemma 30 (*closed equals closure*), Definition 25 (*closure*), and Lemma 29 (*closure is limit of sequences*). □

Definition 32 ***(stationary sequence).*** Let (X,d) be a metric space. A sequence $(x_n)_{n\in\mathbb{N}}$ of X is *stationary* iff

$$\exists N \in \mathbb{N}, \forall n \in \mathbb{N}, \quad n \geq N \quad \Longrightarrow \quad x_n = x_N. \tag{29}$$

N is *a rank* from which the sequence is stationary and x_N is *the stationary value*.

Lemma 33 ***(stationary sequence is convergent).*** *Let (X,d) be a metric space. Let $(x_n)_{n\in\mathbb{N}}$ be a stationary sequence of X. Then, $(x_n)_{n\in\mathbb{N}}$ is convergent with the stationary value as limit.*

Proof. From Definition 32 (*stationary sequence*), let $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $n \geq N$ implies $x_n = x_N$. Let $\varepsilon > 0$. Let $n \in \mathbb{N}$. Assume that $n \geq N$. Then, from Definition 16 (*distance*, d separates points), we have $d(x_n, x_N) = d(x_N, x_N) = 0 \leq \varepsilon$. Hence, from Definition 26 (*convergent sequence*), $(x_n)_{n\in\mathbb{N}}$ is convergent with limit x_N. □

4.2.2 Completeness

Definition 34 ***(Cauchy sequence).*** Let (X,d) be a metric space. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of X. $(x_n)_{n\in\mathbb{N}}$ is a *Cauchy sequence* iff

$$\forall\varepsilon > 0, \exists N \in \mathbb{N}, \forall p, q \in \mathbb{N}, \quad p \geq N \quad \wedge \quad q \geq N \quad \Longrightarrow \quad d(x_p, x_q) \leq \varepsilon. \tag{30}$$

Lemma 35 ***(equivalent definition of Cauchy sequence).*** *Let (X,d) be a metric space. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of X. $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence iff*

$$\forall\varepsilon > 0, \exists N \in \mathbb{N}, \forall p, k \in \mathbb{N}, \quad p \geq N \quad \Longrightarrow \quad d(x_p, x_{p+k}) \leq \varepsilon. \tag{31}$$

Proof. **(30) implies (31).** Assume that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Let $\varepsilon > 0$. From Definition 34 (*Cauchy sequence*), let $N \in \mathbb{N}$ such that for all $p, q \in \mathbb{N}$, $p \geq N$ and $q \geq N$ implies $d(x_p, x_q) \leq \varepsilon$. Let $p, k \in \mathbb{N}$. Assume that $p \geq N$. Then, we also have $q = p + k \geq N$. Thus, $d(x_p, x_{p+k}) = d(x_p, x_q) \leq \varepsilon$.

(31) implies (30). Conversely, assume now that $(x_n)_{n\in\mathbb{N}}$ satisfies (31). Let $\varepsilon > 0$. Then, let $N \in \mathbb{N}$ such that for all $p, k \in \mathbb{N}$, $p \geq N$ implies $d(x_p, x_{p+k}) \leq \varepsilon$. Let $p', q' \in \mathbb{N}$. Assume that $p' \geq N$ and $q' \geq N$. Then, we also have $p = \min(p', q') \geq N$. Let $k = \max(p', q') - p \geq 0$. Then, we have $d(x_p, x_{p+k}) \leq \varepsilon$. Assume that $p' \leq q'$. Then, $p = p'$ and $p + k = q'$. Hence, we have $d(x_p, x_q) = d(x_p, x_{p+k}) \leq \varepsilon$. Conversely, assume that $p' > q'$. Then, $p = q'$ and $p + k = p'$. Hence, from Definition 16 (*distance*, d is symmetric), we have $d(x_p, x_q) = d(x_{p+k}, x_p) = d(x_p, x_{p+k}) \leq \varepsilon$. □

Lemma 36 ***(convergent sequence is Cauchy).*** *Let (X,d) be a metric space. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of X. Assume that $(x_n)_{n\in\mathbb{N}}$ is a convergent sequence. Then, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.*

Proof. Let $\varepsilon > 0$. From Lemma 28 (*limit is unique*), let $l = \lim_{n\to+\infty} \in X$. From Definition 26 (*convergent sequence*), let $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $n \geq N$ implies $d(x_n, l) \leq \frac{\varepsilon}{2}$. Let $p, q \geq N$. Then, from Definition 16 (*distance*, d satisfies triangle inequality and is symmetric), and **field properties of $\mathbb{R}$** we have

$$d(x_p, x_q) \leq d(x_p, l) + d(l, x_q) = d(x_p, l) + d(x_q, l) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, from Definition 34 (*Cauchy sequence*), $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. □

Definition 37 ***(complete subset).*** Let (X,d) be a metric space. A subset Y of X is *complete (for distance d)* iff all Cauchy sequences of Y converge in Y.

Definition 38 ***(complete metric space).*** Let X be a set. Let d be a distance over X. (X, d) is a *complete metric space* iff (X, d) is a metric space and X is complete for distance d.

Lemma 39 ***(closed subset of complete is complete).*** *Let (X, d) be a complete metric space. Let Y be a closed subset of X. Then, Y is complete.*

Proof. Let $(y_n)_{n\in\mathbb{N}}$ be a sequence in Y. Assume that $(y_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Since Y is a subset of X, $(y_n)_{n\in\mathbb{N}}$ is also a Cauchy sequence in X. Then, from Definition 38 (*complete metric space*, $\underline{X \text{ is complete}}$), and Definition 37 (*complete subset*), the sequence $(y_n)_{n\in\mathbb{N}}$ is convergent with limit $y \in X$.

Moreover, from Lemma 29 (*closure is limit of sequences*), the limit a belongs to the closure $\overline{Y}$. Hence, from Lemma 30 (*closed equals closure*, $\underline{Y \text{ is closed}}$), the limit y belongs to Y.

Therefore, from Definition 37 (*complete subset*), Y is complete. □

4.2.3 Continuity

Remark 40. The distance allows the definition of balls centered at each point of a metric space forming neighborhoods for these points. Hence, a metric space can be seen as a topological space.

Remark 41. The distance also allows the definition of entourages making metric spaces specific cases of uniform spaces. Let (X, d) be a metric space. Then, the sets

$$U_r = \{(x, x') \in X \times X \mid d(x, x') \leq r\}$$

for all nonnegative numbers r form a fundamental system of entourages for the standard uniform structure of X. See Theorem 47 below.

Definition 42 ***(continuity in a point).*** Let (X, d_X) and (Y, d_Y) be metric spaces. Let $x \in X$. Let $f : X \to Y$ be a mapping. f is *continuous in* x iff

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x' \in X, \quad d_X(x, x') \leq \delta \quad \Longrightarrow \quad d_Y(f(x), f(x')) \leq \varepsilon. \tag{32}$$

Definition 43 ***(pointwise continuity).*** Let (X, d_X) and (Y, d_Y) be metric spaces. Let $f : X \to Y$ be a mapping. f is *(pointwise) continuous* iff f is continuous in all points of X.

Lemma 44 ***(compatibility of limit with continuous functions).*** *Let (X, d_X) and (Y, d_Y) be metric spaces. Let $f : X \to Y$ be a mapping. Assume that f is pointwise continuous. Then, for all sequence $(x_n)_{n\in\mathbb{N}}$ of X, for all $x \in X$, we have*

$$(x_n)_{n\in\mathbb{N}} \textit{ is convergent with limit } x \quad \Longrightarrow \quad (f(x_n))_{n\in\mathbb{N}} \textit{ is convergent with limit } f(x). \tag{33}$$

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. Let $x \in X$. Assume that $(x_n)_{n\in\mathbb{N}}$ is convergent with limit x. Let $\varepsilon > 0$. Then, from Definition 42 (*continuity in a point*, $\underline{\text{at point } x}$), there exists $\alpha > 0$ such that,

$$\forall n \in \mathbb{N}, \quad d_X(x_n, x) \leq \alpha \quad \Longrightarrow \quad d_Y(f(x_n), f(x)) \leq \varepsilon.$$

And from Definition 26 (*convergent sequence*, $\underline{\text{with } \alpha > 0}$), there exists $N \in \mathbb{N}$ such that,

$$\forall n \in \mathbb{N}, \quad n \geq N \quad \Longrightarrow \quad d_X(x_n, x) \leq \alpha.$$

Thus,

$$\forall n \in \mathbb{N}, \quad n \geq N \quad \Longrightarrow \quad d_Y(f(x_n), f(x)) \leq \varepsilon.$$

Hence, from Definition 26 (*convergent sequence*), the sequence $(f(x_n))_{n\in\mathbb{N}}$ is convergent with limit $f(x)$. □

Definition 45 ***(uniform continuity).*** Let (X, d_X) and (Y, d_Y) be metric spaces. Let $f : X \to Y$ be a mapping. f is *uniformly continuous* iff

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x, x' \in X, \quad d_X(x, x') \leq \delta \quad \Longrightarrow \quad d_Y(f(x), f(x')) \leq \varepsilon. \tag{34}$$

Definition 46 ***(Lipschitz continuity).*** Let (X, d_X) and (Y, d_Y) be metric spaces. Let $f : X \to Y$ be a mapping. Let $k \geq 0$ be a nonnegative number. f is *k-Lipschitz continuous* iff

$$\forall x, x' \in X, \quad d_Y(f(x), f(x')) \leq k\, d_X(x, x'). \tag{35}$$

Then, k is called *Lipschitz constant of f*.

Theorem 47 ***(equivalent definition of Lipschitz continuity).*** *Let (X, d_X) and (Y, d_Y) be metric spaces. Let $f : X \to Y$ be a mapping. Let $k \geq 0$ be a nonnegative number. f is k-*Lipschitz continuous *iff*

$$\forall x, x' \in X, \forall r \geq 0, \quad d_X(x, x') \leq r \quad \Longrightarrow \quad d_Y(f(x), f(x')) \leq kr. \tag{36}$$

Proof. **"Left" implies "right".** Assume that f is k-Lipschitz continuous. Let $x, x' \in X$. Let $r \geq 0$. Assume that $d_X(x, x') \leq r$. Then, from Definition 46 (*Lipschitz continuity*), we have

$$d_Y(f(x), f(x')) \leq k\, d_X(x, x') \leq kr.$$

"Right" implies "left". Conversely, assume now that f satisfies (36). Let $x, x' \in X$. Let $r = d_X(x, x')$. From Definition 16 (*distance*, d_X is nonnegative), r is also nonnegative. From **ordered field properties of** $\mathbb{R}$, we have $d_X(x, x') \leq r$. Hence, from hypothesis, we have

$$d_Y(f(x), f(x')) \leq kr = k\, d_X(x, x').$$

Therefore, from Definition 46 (*Lipschitz continuity*), f is k-Lipschitz continuous. □

Definition 48 ***(contraction).*** Let (X, d) be a metric space. Let $f : X \to Y$ be a mapping. Let $k \geq 0$ be a nonnegative number. f is a *k-contraction* iff f is k-Lipschitz continuous with $k < 1$.

Lemma 49 ***(uniform continuous is continuous).*** *Let (X, d_X) and (Y, d_Y) be metric spaces. Let $f : X \to Y$ be an uniformly continuous mapping. Then, f is continuous.*

Proof. Direct consequence of Definition 45 (*uniform continuity*), Definition 43 (*pointwise continuity*), and Definition 42 (*continuity in a point*). □

Lemma 50 ***(zero-Lipschitz continuous is constant).*** *Let (X, d_X) and (Y, d_Y) be metric spaces. Let $f : X \to Y$ be a 0-Lipschitz continuous mapping. Then, f is constant.*

Proof. Let $x, x' \in X$. Then, from Definition 46 (*Lipschitz continuity*), and Definition 16 (*distance*, d_Y is nonnegative and separates points), we have $d_Y(f(x), f(x')) = 0$ and $f(x) = f(x')$. □

Lemma 51 ***(Lipschitz continuous is uniform continuous).*** *Let (X, d_X) and (Y, d_Y) be metric spaces. Let $f : X \to Y$ be a Lipschitz continuous mapping. Then, f is uniformly continuous.*

Proof. From Definition 46 (*Lipschitz continuity*), let $k \geq 0$ be the Lipschitz constant of f. Let $\varepsilon > 0$ be a positive number.

Case $k = 0$. Then, from Lemma 50 (*zero-Lipschitz continuous is constant*), f is a constant function. Let $\delta = 1 > 0$. Let $x, x' \in X$. Assume that $d_X(x, x') < \delta$. Then, we have

$d_Y(f(x), f(x')) = 0 < \varepsilon$. Hence, from Definition 45 (*uniform continuity*), f is uniformly continuous.

Case $k \neq 0$. Then, $k > 0$. From **ordered field properties of $\mathbb{R}$**, let $\delta = \frac{\varepsilon}{k} > 0$. Let $x, x' \in X$. Assume that $d_X(x, x') \leq \delta$. Then, from **ordered field properties of $\mathbb{R}$**, we have

$$d_Y(f(x), f(x')) \leq k\, d_X(x, x') \leq k\delta = \varepsilon.$$

Hence, from Definition 45 (*uniform continuity*), f is uniformly continuous. □

4.2.4 Fixed point theorem

Definition 52 *(iterated function sequence).* Let (X, d) be a metric space. Let $f : X \to X$ be a mapping. An *iterated function sequence associated with* f is a sequence of X defined by

$$x_0 \in X \quad \wedge \quad \forall n \in \mathbb{N}, \quad x_{n+1} = f(x_n). \tag{37}$$

Lemma 53 *(stationary iterated function sequence).* *Let (X, d) be a metric space. Let $f : X \to X$ be a mapping. Let $(x_n)_{n\in\mathbb{N}}$ be an iterated function sequence associated with f such that*

$$\exists N \in \mathbb{N}, \quad x_{N+1} = f(x_N) = x_N. \tag{38}$$

Then, the sequence $(x_n)_{n\in\mathbb{N}}$ is stationary.

Proof. For $i \in \mathbb{N}$, let $P(i)$ be the property $x_{N+i+1} = x_N$.

Induction: $P(0)$. Property $P(0)$ holds by hypothesis.

Induction: $P(i)$ implies $P(i+1)$. Let $i \in \mathbb{N}$. Assume that $P(i)$ holds. Then, from Definition 52 (*iterated function sequence*), and hypothesis, we have

$$x_{N+i+2} = f(x_{N+i+1}) = f(x_N) = x_N.$$

Hence, $P(i+1)$ holds.

Therefore, by induction on $i \in \mathbb{N}$, we have, for all $i \in \mathbb{N}$, $P(i)$, and from Definition 32 (*stationary sequence*), the sequence $(x_n)_{n\in\mathbb{N}}$ is stationary. □

Lemma 54 *(iterate Lipschitz continuous mapping).* *Let (X, d) be a metric space. Let $k \geq 0$. Let $f : X \to X$ be a k-Lipschitz continuous mapping. Let $(x_n)_{n\in\mathbb{N}}$ be an iterated function sequence associated with f. Then,*

$$\forall n \in \mathbb{N}, \quad d(x_n, x_{n+1}) \leq k^n\, d(x_0, x_1). \tag{39}$$

Proof. For $n \in \mathbb{N}$, let $P(n)$ be the property $d(x_n, x_{n+1}) \leq k^n\, d(x_0, x_1)$.

Induction: $P(0)$. Property $P(0)$ is a direct consequence of convention $0^0 = 1$ and **ordered field properties of $\mathbb{R}$**.

Induction: $P(n)$ implies $P(n+1)$. Let $n \in \mathbb{N}$ be a natural number. Assume that $P(n)$ holds. Then, from Definition 52 (*iterated function sequence*), Definition 46 (*Lipschitz continuity*), **field properties of $\mathbb{R}$**, and hypotheses, we have

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &= d(f(x_n), f(x_{n+1})) \\ &\leq k\, d(x_n, x_{n+1}) \\ &\leq k\, k^n\, d(x_0, x_1) \\ &= k^{n+1}\, d(x_0, x_1). \end{aligned}$$

Hence, $P(n+1)$ holds.

Therefore, by induction on $n \in \mathbb{N}$, we have, for all $n \in \mathbb{N}$, $P(n)$. □

Lemma 55 ***(convergent iterated function sequence).*** *Let (X, d) be a metric space. Let $f : X \to X$ be a Lipschitz continuous mapping. Let $(x_n)_{n\in\mathbb{N}}$ be a convergent iterated function sequence associated with f. Then, the limit of the sequence is a fixed point of f.*

Proof. From Definition 46 (*Lipschitz continuity*), let $k \geq 0$ be the Lipschitz constant of f. From Definition 26 (*convergent sequence*), let $a = \lim_{n\to+\infty} x_n \in X$ be the limit of the sequence.

Case $k = 0$. Then, from Lemma 50 (*zero-Lipschitz continuous is constant*), f is constant of value $f(a)$. Thus, from Definition 32 (*stationary sequence*), the sequence $(x_n)_{n\in\mathbb{N}}$ is stationary from rank 1. Hence, from Lemma 33 (*stationary sequence is convergent*), $(x_n)_{n\in\mathbb{N}}$ is convergent with limit $f(a)$.

Case $k \neq 0$. Then, from Definition 46 (*Lipschitz continuity*), we have $k > 0$. Let $\varepsilon > 0$. From Definition 26 (*convergent sequence*), let $N \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$, $n \geq N$ implies $d(x_n, a) \leq \frac{\varepsilon}{k}$. Let $N' = N + 1$. Let $n \in \mathbb{N}$. Assume that $n \geq N'$. Then, $n - 1 \geq N$. Thus, from Definition 52 (*iterated function sequence*), Definition 46 (*Lipschitz continuity*), and **ordered field properties of $\mathbb{R}$**, we have

$$d(x_n, f(a)) = d(f(x_{n-1}), f(a)) \leq k\, d(x_{n-1}, a) \leq \varepsilon.$$

Hence, from Definition 26 (*convergent sequence*), the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent with limit $f(a)$.

Therefore, in both cases, from Lemma 28 (*limit is unique*), $f(a) = a$. ☐

Theorem 56 ***(fixed point).*** *Let (X, d) be a complete metric space. Let $f : X \to X$ be a contraction. Then, there exists a unique fixed point $a \in X$ such that $f(a) = a$. Moreover, all iterated function sequences associated with f are convergent with limit a.*

Proof. **Uniqueness.** Let $a, a' \in X$ be two fixed points of f. Then, from Definition 48 (*contraction*), and Definition 46 (*Lipschitz continuity*), we have $d(a, a') = d(f(a), f(a')) \leq k\, d(a, a')$. Thus, from **ordered field properties of $\mathbb{R}$**, Definition 48 (*contraction*, $\underline{k < 1}$), and Definition 16 (*distance*, $\underline{d \text{ is nonnegative}}$), we have $0 \leq (1 - k)\, d(a, a') \leq 0$. Therefore, from the zero-product property of $\mathbb{R}$, Definition 48 (*contraction*, $\underline{k \neq 1}$), and Definition 16 (*distance*, $\underline{d \text{ separates points}}$), we have $a = a'$.

Convergence of iterated function sequences and existence. Let $x_0 \in X$. Let $(x_n)_{n\in\mathbb{N}}$ be an iterated function sequence associated with f. Let $p, m \in \mathbb{N}$. Then, from Lemma 18 (*iterated triangle inequality*), Lemma 54 (*iterate Lipschitz continuous mapping*), **field properties of $\mathbb{R}$**, **the formula for the sum of the first terms of a geometric series**, and Definition 48 (*contraction*, $\underline{0 \leq k < 1}$), we have

$$\begin{aligned} d(x_p, x_{p+m}) &\leq \sum_{i=0}^{m-1} d(x_{p+i}, x_{p+i+1}) \\ &\leq \left(\sum_{i=0}^{m-1} k^{p+i} \right) d(x_0, x_1) \\ &= k^p \frac{1 - k^m}{1 - k} d(x_0, x_1) \\ &\leq \frac{k^p}{1 - k} d(x_0, x_1). \end{aligned}$$

Case $k = 0$. Then, from Lemma 50 (*zero-Lipschitz continuous is constant*), f is constant. Let $a = f(x_0)$. Then, for all $x \in X$, $f(x) = a$. In particular, $f(a) = a$, and for all $n \in \mathbb{N}$, $x_{n+1} = f(x_n) = a$. Thus, from Definition 32 (*stationary sequence*), the sequence $(x_n)_{n\in\mathbb{N}}$ is stationary from rank 1 with stationary value a.

Case $x_1 = x_0$. Then, from Lemma 53 (*stationary iterated function sequence*), the sequence $(x_n)_{n\in\mathbb{N}}$ is stationary from rank 0 with stationary value $x_0 = x_1 = f(x_0) = a$.

Hence, in both cases, from Lemma 33 (*stationary sequence is convergent*), the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent with limit a.

Case $k \neq 0$ and $x_1 \neq x_0$. Then, from Definition 48 (*contraction*, $\underline{0 \leq k < 1}$), and Definition 16 (*distance*, $\underline{d \text{ separates points, contrapositive}}$), we have $0 < k < 1$ and $d(x_0, x_1) \neq 0$. Let $\varepsilon > 0$. Let

$$\zeta = \frac{(1-k)\varepsilon}{d(x_0,x_1)} > 0, \quad \xi = \max\left(0, \frac{\ln \zeta}{\ln k}\right) \geq 0, \quad N = \lceil \xi \rceil \in \mathbb{N}.$$

Let $p, m \in \mathbb{N}$. Assume that $p \geq N$. Then, from **the definition of ceiling and max functions**, we have $p \geq \xi \geq \frac{\ln \zeta}{\ln k}$. Thus, from **ordered field properties of $\mathbb{R}$ ($\ln k$ is negative)**, and **increase of the exponential function**, we have $p \ln k \leq \ln \zeta$, hence $k^p \leq \zeta$, and finally

$$\frac{k^p}{1-k} d(x_0, x_1) \leq \varepsilon.$$

Hence, from Lemma 35 (*equivalent definition of Cauchy sequence*), Definition 38 (*complete metric space*), and Definition 37 (*complete subset*), $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence that is convergent with limit $a \in X$.

Therefore, in all cases, from Lemma 55 (*convergent iterated function sequence*), a is a fixed point of f. □

4.3 Vector space

Remark 57. Statements and proofs are presented in the case of vector spaces over the field of real numbers $\mathbb{R}$, but most can be generalized with minor or no alteration to the case of vector spaces over the field of complex numbers $\mathbb{C}$. When the same statement holds for both cases, the field is denoted $\mathbb{K}$. Note that in both cases, $\mathbb{R} \subset \mathbb{K}$.

4.3.1 Basic notions and notations

Definition 58 *(vector space).* Let E be a set equipped with two vector operations: an addition $(+ : E \times E \to E)$ and a scalar multiplication $(\cdot : \mathbb{K} \times E \to E)$. $(E, +, \cdot)$ is a *vector space over field* $\mathbb{K}$, or simply E is a *space*, iff $(E, +)$ is an abelian group with identity element 0_E (zero vector), and scalar multiplication is distributive wrt vector addition and field addition, compatible with field multiplication, and admits $1_\mathbb{K}$ as identity element (simply denoted 1):

$$\forall \lambda \in \mathbb{K}, \forall u, v \in E, \qquad \lambda \cdot (u + v) = \lambda \cdot u + \lambda \cdot v; \tag{40}$$
$$\forall \lambda, \mu \in \mathbb{K}, \forall u \in E, \qquad (\lambda + \mu) \cdot u = \lambda \cdot u + \mu \cdot u; \tag{41}$$
$$\forall \lambda, \mu \in \mathbb{K}, \forall u \in E, \qquad \lambda \cdot (\mu \cdot u) = (\lambda\mu) \cdot u; \tag{42}$$
$$\forall u \in E, \qquad 1 \cdot u = u. \tag{43}$$

Remark 59. The $\cdot$ infix sign in the scalar multiplication may be omitted.

Remark 60. Vector spaces over $\mathbb{R}$ are called *real* spaces, and vector spaces over $\mathbb{C}$ are called *complex* spaces.

Definition 61 *(set of mappings to space).* Let X be a set. Let E be a space. The *set of mappings from X to E* is denoted $\mathcal{F}(X, E)$.

Definition 62 *(linear map).* Let $(E, +_E, \cdot_E)$ and $(F, +_F, \cdot_F)$ be spaces. A mapping $f : E \to F$ is a *linear map from E to F* iff it preserves vector operations, i.e. iff it is additive and homogeneous of degree 1:

$$\forall u, v \in E, \qquad f(u +_E v) = f(u) +_F f(v); \tag{44}$$
$$\forall \lambda \in \mathbb{K}, \forall u \in E, \qquad f(\lambda \cdot_E u) = \lambda \cdot_F f(u). \tag{45}$$

Definition 63 ***(set of linear maps).*** Let E, F be spaces. The *set of linear maps from E to F* is denoted $\mathcal{L}(E, F)$.

Definition 64 ***(linear form).*** Let E be a vector space over field $\mathbb{K}$. A *linear form on E* is a linear map from E to $\mathbb{K}$.

Definition 65 ***(bilinear map).*** Let $(E, +_E, \cdot_E)$, $(F, +_F, \cdot_F)$ and $(G, +_G, \cdot_G)$ be spaces. A mapping $\varphi : E \times F \to G$ is a *bilinear map from $E \times F$ to G* iff it is left additive, right additive, and left and right homogeneous of degree 1:

$$\forall u, v \in E, \forall w \in F, \qquad \varphi(u +_E v, w) = \varphi(u, w) +_G \varphi(v, w); \tag{46}$$
$$\forall u \in E, \forall v, w \in F, \qquad \varphi(u, v +_F w) = \varphi(u, v) +_G \varphi(u, w); \tag{47}$$
$$\forall \lambda \in \mathbb{K}, \forall u \in E, \forall v \in F, \qquad \varphi(\lambda \cdot_E u, v) = \lambda \cdot_G \varphi(u, v) = \varphi(u, \lambda \cdot_F v). \tag{48}$$

Definition 66 ***(bilinear form).*** Let E be a space. A *bilinear form on E* is a bilinear map from $E \times E$ to $\mathbb{K}$.

Definition 67 ***(set of bilinear forms).*** Let E be a space. The *set of bilinear forms on E* is denoted $\mathcal{L}_2(E) = \mathcal{L}(E \times E, \mathbb{K})$.

4.3.2 Linear algebra

Lemma 68 ***(zero times yields zero).*** *Let $(E, +, \cdot)$ be a space. Then,*

$$\forall u \in E, \quad 0 \cdot u = 0_E. \tag{49}$$

Proof. Let $u \in E$ be a vector. From Definition 58 (*vector space*, (E, +) is an abelian group, scalar multiplication admits 1 as identity element and is distributive wrt field addition), and **field properties of $\mathbb{K}$**, we have

$$0 \cdot u = 0 \cdot u + u + (-u) = 0 \cdot u + 1 \cdot u + (-u) = (0 + 1) \cdot u + (-u) = 1 \cdot u + (-u) = u + (-u) = 0_E$$

□

Lemma 69 ***(minus times yields opposite vector).*** *Let $(E, +, \cdot)$ be a space. Then,*

$$\forall \lambda \in \mathbb{K}, \forall u \in E, \quad (-\lambda) \cdot u = -(\lambda \cdot u). \tag{50}$$

Proof. Let $\lambda \in \mathbb{K}$ be a scalar. Let $u \in E$ be a vector. From Definition 58 (*vector space*, scalar multiplication is distributive wrt field addition), **field properties of $\mathbb{K}$**, and Lemma 68 (*zero times yields zero*), we have

$$\lambda \cdot u + (-\lambda) \cdot u = (\lambda - \lambda) \cdot u = 0 \cdot u = 0_E.$$

Therefore, from Definition 58 (*vector space*, (E, +) is an abelian group), $(-\lambda) \cdot u$ is the opposite of $\lambda \cdot u$. □

Definition 70 ***(vector subtraction).*** Let $(E, +, \cdot)$ be a space. *Vector subtraction*, denoted by the infix operator $-$, is defined by

$$\forall u, v \in E, \quad u - v = u + (-v). \tag{51}$$

Definition 71 ***(scalar division).*** Let $(E, +, \cdot)$ be a space. *Scalar division*, denoted by the infix operator $/$, is defined by

$$\forall \lambda \in \mathbb{K}^\star, \forall u \in E, \quad \frac{u}{\lambda} = \frac{1}{\lambda} \cdot u. \tag{52}$$

Lemma 72 ***(times zero yields zero).*** *Let $(E, +, \cdot)$ be a space. Then,*

$$\forall \lambda \in \mathbb{K}, \quad \lambda \cdot 0_E = 0_E. \tag{53}$$

Proof. Let $\lambda \in \mathbb{K}$ be a scalar. From Definition 58 (*vector space*, $(E,+)$ is an abelian group and scalar multiplication is distributive wrt vector addition), and Definition 70 (*vector subtraction*), we have

$$\lambda \cdot 0_E = \lambda \cdot (0_E - 0_E) = \lambda \cdot 0_E - \lambda \cdot 0_E = 0_E.$$

□

Lemma 73 ***(zero-product property).*** *Let $(E, +, \cdot)$ be a space. Then,*

$$\forall \lambda \in \mathbb{K}, \forall u \in E, \quad \lambda \cdot u = 0_E \quad \Longleftrightarrow \quad \lambda = 0 \quad \vee \quad u = 0_E. \tag{54}$$

Proof. Let $\lambda \in \mathbb{K}$ be a scalar. Let $u \in E$ be a vector.

"Left" implies "right". Assume that $\lambda = 0$ or $u = 0_E$ Then, from Lemma 68 (*zero times yields zero*), and Lemma 72 (*times zero yields zero*), we have $\lambda \cdot u = 0_E$.

"Right" implies "left". Assume that $\lambda \cdot u = 0_E$ and $\lambda \neq 0$ Then, from Definition 58 (*vector space*, scalar multiplication admits 1 as identity element and is compatible with field multiplication), **field properties of $\mathbb{K}$**, and Lemma 72 (*times zero yields zero*), we have

$$u = 1 \cdot u = \left(\frac{1}{\lambda}\lambda\right) \cdot u = \frac{1}{\lambda} \cdot (\lambda \cdot u) = \frac{1}{\lambda} \cdot 0_E = 0_E.$$

Hence, since $\boldsymbol{(P \wedge \neg Q \Rightarrow R) \Leftrightarrow (P \Rightarrow Q \vee R)}$, $\lambda \cdot u = 0_E$ implies $\lambda = 0$ or $u = 0_E$. □

Definition 74 ***(subspace).*** Let $(E, +, \cdot)$ be a space. Let $F \subset E$ be a subset of E. Let $+_{|F}$ be the restrictions of $+$ to $F \times F$. Let $\cdot_{|F}$ be the restrictions of $\cdot$ to $\mathbb{K} \times F$. F is a *vector subspace of E*, or simply a *subspace of E*, iff $(F, +_{|F}, \cdot_{|F})$ is a space.

Remark 75. In particular, a subspace is closed under restricted vector operations.

Remark 76. Usually, restrictions $+_{|F}$ and $\cdot_{|F}$ are still denoted $+$ and $\cdot$.

Lemma 77 ***(trivial subspaces).*** *Let E be a space. Then, E and $\{0_E\}$ are subspaces of E.*

Proof. E and $\{0_E\}$ are trivially subsets of E. E is a space. $\{0_E\}$ is trivially a space. Therefore, from Definition 74 (*subspace*), E and $\{0_E\}$ are subspaces of E. □

Lemma 78 ***(closed under vector operations is subspace).*** *Let E be a space. Let F be a subset of E. F is a subspace of E iff $0_E \in F$ and F is closed under vector addition and scalar multiplication:*

$$\forall u, v \in F, \qquad u + v \in F; \tag{55}$$
$$\forall \lambda \in \mathbb{K}, \forall u \in F, \qquad \lambda u \in F. \tag{56}$$

Proof. **"If".** Assume that F contains 0_E and is closed under vector addition and scalar multiplication. Then, F is closed under the restriction to F of vector operations. Let $u, v \in F$ be vectors. Then, from Lemma 69 (*minus times yields opposite vector*, with $\lambda = 1$), $-v = (-1)v$ belongs to F, and $u - v = u + (-v)$ also belongs to F. Thus, from **group theory**, $(F, +_{|F})$ is a subgroup of $(E, +)$. Hence, from Definition 58 (*vector space*, $(E, +)$ is an abelian group), and **group theory**, $(F, +_{|F})$ is also an abelian group and $0_F = 0_E$. Since F is a subset of E, and E is a space, properties (40) to (43) are trivially satisfied over F. Therefore, from Definition 74 (*subspace*), F is a subspace of E.

"Only if". Conversely, assume now that F is a subspace of E. Then, from Definition 74 (*subspace*, F is a space), and Definition 58 (*vector space*, $(F, +_{|F})$ is an abelian group), F contains $0_F = 0_E$ and F is closed under the restriction to F of vector operations. Therefore, F is closed under vector addition and scalar multiplication. □

Lemma 79 ***(closed under linear combination is subspace).*** *Let E be a space. Let F be a subset of E. F is a subspace of E iff $0_E \in F$ and F is closed under linear combination:*

$$\forall \lambda, \mu \in \mathbb{K}, \forall u, v \in F, \quad \lambda u + \mu v \in F. \tag{57}$$

Proof. **"If".** Assume that F contains 0_E and is closed under linear combination. Let $u, v \in F$ be vectors. Let $\lambda \in \mathbb{K}$ be a scalar. Then, from Definition 58 (*vector space*, scalar multiplication in E admits 1 as identity element), $u + v = 1u + 1v$ belongs to F, and from Lemma 73 (*zero-product property*), and Definition 58 (*vector space*, $(E, +)$ is an abelian group), $\lambda u = \lambda u + 0 \cdot 0_E$ belongs to F. Thus, F contains 0_E and is closed under vector operations. Therefore, from Lemma 78 (*closed under vector operations is subspace*), F is a subspace of E.

"Only if". Conversely, assume now that F is a subspace of E. Let $\lambda, \mu \in \mathbb{K}$ be scalars. Let $u, v \in F$ be vectors. Then, from Lemma 78 (*closed under vector operations is subspace*), F contains 0_E, F is closed under scalar multiplication, hence $u' = \lambda u$ and $v' = \mu v$ belong to F, and F is closed by vector addition, hence $u' + v' = \lambda u + \mu v$ belongs to F. Therefore, F is closed under linear combination. □

Definition 80 ***(linear span).*** Let E be a space. Let $u \in E$ be a vector. The *linear span of u*, denoted $\mathrm{span}(\{u\})$, is defined by

$$\mathrm{span}(\{u\}) = \{\lambda u \,|\, \lambda \in \mathbb{K}\}. \tag{58}$$

Definition 81 ***(sum of subspaces).*** Let E be a space. Let F, F' be subspaces of E. The *sum of F and F'* is the subset of E defined by

$$F + F' = \{u + u' \,|\, u \in F,\ u' \in F'\}. \tag{59}$$

Definition 82 ***(finite dimensional subspace).*** Let E be a space. Let F be a subspace of E. F *is a finite dimensional subspace* iff there exists $n \in \mathbb{N}$, and $u_1, \ldots, u_n \in E$ such that

$$\begin{aligned} F &= \mathrm{span}(\{u_1, \ldots, u_n\}) = \mathrm{span}(\{u_1\}) + \ldots + \mathrm{span}(\{u_n\}) \\ &= \{\lambda_1 u_1 + \ldots + \lambda_n u_n \,|\, \lambda_1, \ldots, \lambda_n \in \mathbb{K}\}. \end{aligned} \tag{60}$$

Definition 83 ***(direct sum of subspaces).*** Let E be a space. Let F, F' be subspaces of E. The sum $F + F'$ is called *direct sum*, and it is denoted $F \oplus F'$, iff all vectors of the sum admit a unique decomposition:

$$\forall u, v \in F, \forall u', v' \in F', \quad u + u' = v + v' \quad \Longrightarrow \quad u = v \quad \wedge \quad u' = v'. \tag{61}$$

Lemma 84 ***(equivalent definitions of direct sum).*** *Let E be a space. Let F, F' be subspaces of E. The sum $F + F'$ is direct iff one of the following equivalent properties is satisfied:*

$$F \cap F' = \{0_E\}; \tag{62}$$

$$\forall u \in F, \forall u' \in F', \quad u + u' = 0_E \quad \Longrightarrow \quad u = u' = 0_E. \tag{63}$$

Proof. **(61) implies (62).** Assume that the sum $F + F'$ is direct. Let $v \in F \cap F'$ be a vector in the intersection. Then, from Definition 58 (*vector space*, $(E, +)$ is an abelian group), v admits two decompositions, $v = v + 0_E = 0_E + v$. Thus, from Definition 83 (*direct sum of subspaces*), $v = 0_E$.

(62) implies (63). Assume now that $F \cap F' = \{0_E\}$. Let $u \in F$ and $u' \in F'$ be vectors. Assume that $u + u' = 0_E$. Then, from Lemma 69 (*minus times yields opposite vector*, with $\lambda = 1$), and Lemma 78 (*closed under vector operations is subspace*, scalar multiplication), $u = -u' = (-1)u'$ belongs to F' and $u' = -u = (-1)u$ belongs to F. Hence, $u, u' \in F \cap F'$, and $u = u' = 0_E$.

(63) implies (61). Assume finally that 0_E admits a unique decomposition. Let $u, v \in F$ and $u', v' \in F'$ be vectors. Assume that $u + u' = v + v'$. Then, from Definition 58 (*vector space*, $(E, +)$ is an abelian group), and Definition 70 (*vector subtraction*), we have $(u - v) + (u' - v') = 0_E$. Thus, from hypothesis, we have $u - v = u' - v' = 0_E$, and from Definition 58 (*vector space*, $(E, +)$ is an abelian group), we have $u = v$ and $u' = v'$.

Therefore, all three properties are equivalent. □

Lemma 85 ***(direct sum with linear span).*** *Let E be a space. Let F be a subspace of E. Let $u \in E$ be a vector. Assume that $u \notin F$. Then, the sum $F + \mathrm{span}(\{u\})$ is direct.*

Proof. From Lemma 78 (*closed under vector operations is subspace*, F and $\mathrm{span}(\{u\})$ are subspace), we have $0_E \in F$ and $0_E \in \mathrm{span}(\{u\})$. Let $v \in F \cap \mathrm{span}(\{u\})$. Assume that $v \neq 0_E$. Then, from Definition 80 (*linear span*), let $\lambda \in \mathbb{K}$ such that $v = \lambda u$. Thus, from hypothesis, and Lemma 73 (*zero-product property*, contrapositive), we have $\lambda \neq 0_{\mathbb{K}}$. Hence, from **field properties of $\mathbb{K}$**, and Lemma 78 (*closed under vector operations is subspace*), we have $\frac{1}{\lambda} v = u \in F$. Which is impossible by hypothesis.

Therefore, from Lemma 84 (*equivalent definitions of direct sum*), the sum $F + \mathrm{span}(\{u\})$ is direct. □

Definition 86 ***(product vector operations).*** Let $(E, +_E, \cdot_E)$ and $(F, +_F, \cdot_F)$ be spaces. The *product vector operations induced on* $E \times F$ are the mappings $+_{E\times F} : (E \times F) \times (E \times F) \to E \times F$ and $\cdot_{E\times F} : \mathbb{K} \times (E \times F) \to E \times F$ defined by

$$\forall (u, v), (u', v') \in E \times F, \qquad (u, v) +_{E\times F} (u', v') = (u +_E u', v +_F v'); \tag{64}$$

$$\forall \lambda \in \mathbb{K}, \forall (u, v) \in E \times F, \qquad \lambda \cdot_{E\times F} (u, v) = (\lambda \cdot_E u, \lambda \cdot_F v). \tag{65}$$

Lemma 87 ***(product is space).*** *Let $(E, +_E, \cdot_E)$ and $(F, +_F, \cdot_F)$ be spaces. Let $+_{E\times F}$ and $\cdot_{E\times F}$ be the product vector operations induced on $E \times F$. Then, $(E \times F, +_{E\times F}, \cdot_{E\times F})$ is a space.*

Proof. From **group theory**, $(E \times F, +_{E\times F})$ is an abelian group with identity element $0_{E\times F} = (0_E, 0_F)$. Distributivity of the product scalar multiplication wrt product vector addition and field addition, compatibility of the product scalar multiplication with field multiplication, and 1 is the identity element for the product scalar multiplication are direct consequences of Definition 58 (*vector space*), and Definition 86 (*product vector operations*).

Therefore, from Definition 58 (*vector space*), $(E \times F, +_{E\times F}, \cdot_{E\times F})$ is a space. □

Definition 88 ***(inherited vector operations).*** Let X be a set. Let $(E, +_E, \cdot_E)$ be a space. The *vector operations inherited on* $\mathcal{F}(X, E)$ are the mappings $+_{\mathcal{F}(X,E)} : \mathcal{F}(X, E) \times \mathcal{F}(X, E) \to \mathcal{F}(X, E)$ and $\cdot_{\mathcal{F}(X,E)} : \mathbb{K} \times \mathcal{F}(X, E) \to \mathcal{F}(X, E)$ defined by

$$\forall f, g \in \mathcal{F}(X, E), \forall x \in X, \qquad (f +_{\mathcal{F}(X,E)} g)(x) = f(x) +_E g(x); \tag{66}$$

$$\forall \lambda \in \mathbb{K}, \forall f \in \mathcal{F}(X, E), \forall x \in X, \qquad (\lambda \cdot_{\mathcal{F}(X,E)} f)(x) = \lambda \cdot_E f(x). \tag{67}$$

Remark 89. Usually, inherited vector operations are denoted the same way as the vector operations of the target space.

Lemma 90 ***(space of mappings to a space).*** *Let X be a set. Let $(E, +_E, \cdot_E)$ be a space. Let $+_{\mathcal{F}(X,E)}$ and $\cdot_{\mathcal{F}(X,E)}$ be the vector operations inherited on $\mathcal{F}(X, E)$. Then, $(\mathcal{F}(X, E), +_{\mathcal{F}(X,E)}, \cdot_{\mathcal{F}(X,E)})$ is a space.*

Proof. From Definition 61 (*set of mappings to space*), and **group theory**, $(\mathcal{F}(X,E), +_{\mathcal{F}(X,E)})$ is an abelian group with identity element

$$0_{\mathcal{F}(X,E)} = (x \in X \mapsto 0_E).$$

Distributivity of the inherited scalar multiplication wrt inherited vector addition and field addition, compatibility of the inherited scalar multiplication with field multiplication, and 1 is the identity element for the inherited scalar multiplication are direct consequences of Definition 58 (*vector space*), and Definition 88 (*inherited vector operations*).

Therefore, from Definition 58 (*vector space*), $(\mathcal{F}(X,E), +_{\mathcal{F}(X,E)}, \cdot_{\mathcal{F}(X,E)})$ is a space. □

Lemma 91 ***(linear map preserves zero).*** *Let E and F be spaces. Let f be a linear map from E to F. Then, $f(0_E) = 0_F$.*

Proof. From Lemma 68 (*zero times yields zero*), and Definition 62 (*linear map*, f is homogeneous of degree 1), we have

$$f(0_E) = f(0_{\mathbb{K}} \cdot_E 0_E) = 0_{\mathbb{K}} \cdot_F f(0_E) = 0_F.$$

□

Lemma 92 ***(linear map preserves linear combinations).*** *Let $(E, +_E, \cdot_E)$ and $(F, +_F, \cdot_F)$ be spaces. Let $f : E \to F$ be a mapping from E to F. Then, f is a linear map from E to F iff it preserves linear combinations:*

$$\forall \lambda, \mu \in \mathbb{K}, \forall u, v \in E, \quad f(\lambda \cdot_E u +_E \mu \cdot_E v) = \lambda \cdot_F f(u) +_F \mu \cdot_F f(v). \tag{68}$$

Proof. **"If".** Assume that (68) holds. Let $u, v \in E$ be vectors. Then, from Definition 58 (*vector space*, scalar multiplications in E and F admit 1 as identity element), we have

$$f(u +_E v) = f(1 \cdot_E u +_E 1 \cdot_E v) = 1 \cdot_F f(u) +_F 1 \cdot_F f(v) = f(u) +_F f(v).$$

Hence, f is additive. Let $\lambda \in \mathbb{K}$ be a scalar. Let u be a vector. Then, from Lemma 68 (*zero times yields zero*, in E and F), and Definition 58 (*vector space*, $(E, +_E)$ and $(F, +_F)$ are abelian groups), we have

$$f(\lambda \cdot_E u) = f(\lambda \cdot_E u +_E 0 \cdot_E 0_E) = \lambda \cdot_F f(u) +_F 0 \cdot_F f(0_E) = \lambda \cdot_F f(u).$$

Hence, f is homogeneous of degree 1. Therefore, from Definition 62 (*linear map*), f is a linear map from E to F.

"Only if". Conversely, assume now that f is a linear map from E to F. Let $\lambda, \mu \in \mathbb{K}$ be scalars. Let $u, v \in E$ be vectors. Then, from Definition 62 (*linear map*, f is additive), $f(\lambda \cdot_E u +_E \mu \cdot_E v) = f(\lambda \cdot_E u) +_F f(\mu \cdot_E v)$, and ($f$ is homogeneous of degree 1) $f(\lambda \cdot_E u) = \lambda \cdot_F f(u)$ and $f(\mu \cdot_E v) = \mu \cdot_F f(v)$. Hence, we have

$$f(\lambda \cdot_E u +_E \mu \cdot_E v) = \lambda \cdot_F f(u) +_F \mu \cdot_F f(v).$$

□

Lemma 93 ***(space of linear maps).*** *Let E and $(F, +_F, \cdot_F)$ be spaces. Let $+_{\mathcal{F}(E,F)}$ and $\cdot_{\mathcal{F}(E,F)}$ be the vector operations inherited on $\mathcal{F}(E,F)$. Then, $\mathcal{L}(E,F)$ is a subspace of $(\mathcal{F}(E,F), +_{\mathcal{F}(E,F)}, \cdot_{\mathcal{F}(E,F)})$.*

Proof. From Definition 63 (*set of linear maps*), and Lemma 90 (*space of mappings to a space*), $\mathcal{L}(E,F)$ is a subset of $\mathcal{F}(E,F)$. From Lemma 72 (*times zero yields zero*), and Definition 58 (*vector space*, $(F, +_F)$ is an abelian group), $0_{\mathcal{F}(E,F)}$ trivially preserves vector operations. Hence, from Definition 62 (*linear map*), $0_{\mathcal{F}(E,F)}$ is a linear map from E to F. From Definition 88 (*inherited vector operations*), Definition 62 (*linear map*), and Definition 58 (*vector space*, $(E, +_E)$ and $(F, +_F)$ are abelian groups and scalar multiplications in E and F are compatible with field multiplication), $\mathcal{L}(E,F)$ is trivially closed under linear combination.

Therefore, from Lemma 79 (*closed under linear combination is subspace*), $\mathcal{L}(E,F)$ is a subspace of $\mathcal{F}(E,F)$. □

Definition 94 ***(identity map).*** Let E be a space. The *identity map on* E is the mapping $\mathrm{Id}_E : E \to E$ defined by

$$\forall u \in E, \quad \mathrm{Id}_E(u) = u. \tag{69}$$

Lemma 95 ***(identity map is linear map).*** *Let E be a space. Then, the identity map Id_E is a linear map.*

Proof. Direct consequence of Definition 94 (*identity map*), and Definition 62 (*linear map*). □

Lemma 96 ***(composition of linear maps is bilinear).*** *Let E, F, G be spaces. Then, the composition of functions is a bilinear map from $\mathcal{L}(E, F) \times \mathcal{L}(F, G)$ to $\mathcal{L}(E, G)$.*

Proof. From Lemma 93 (*space of linear maps*, $\mathcal{L}(E, F)$, $\mathcal{L}(F, G)$ and $\mathcal{L}(E, G)$ are spaces), and Lemma 87 (*product is space*), $\mathcal{L}(E, F) \times \mathcal{L}(F, G)$ is a space.

Let $f \in \mathcal{L}(E, F)$ and $g \in \mathcal{L}(F, G)$ be linear maps. Let $\lambda, \mu \in \mathbb{K}$ be scalars. Let $u, v \in E$ be vectors. Then, from **the definition of composition of functions**, and Lemma 92 (*linear map preserves linear combinations*, for f and g), we have

$$\begin{aligned}
(g \circ f)(\lambda u + \mu v) &= g(f(\lambda u + \mu v)) \\
&= g(\lambda f(u) + \mu f(v)) \\
&= \lambda g(f(u)) + \mu g(f(v)) \\
&= \lambda (g \circ f)(u) + \mu (g \circ f)(v).
\end{aligned}$$

Hence, $g \circ f$ belongs to $\mathcal{L}(E, G)$, and composition is a mapping from space $\mathcal{L}(E, F) \times \mathcal{L}(F, G)$ to space $\mathcal{L}(E, G)$.

From **the definition of composition of functions**, and Definition 88 (*inherited vector operations*), composition of linear maps is trivially left additive and left homogeneous of degree 1. From **the definition of composition of functions**, Definition 88 (*inherited vector operations*), and Definition 62 (*linear map*, left argument "g" is additive and homogeneous of degree 1), composition of linear maps is trivially right additive and right homogeneous of degree 1.

Therefore, from Definition 65 (*bilinear map*), composition of linear maps is a bilinear map from $\mathcal{L}(E, F) \times \mathcal{L}(F, G)$ to $\mathcal{L}(E, G)$. □

Definition 97 ***(isomorphism).*** Let E and F be spaces. An *isomorphism from E onto F* is a linear map from E to F that is bijective.

Definition 98 ***(kernel).*** Let E and F be spaces. Let $f \in \mathcal{L}(E, F)$ be a linear map from E to F. The *kernel of f* (or *null space of f*), denoted $\ker(f)$, is the subset of E defined by

$$\ker(f) = \{u \in E \mid f(u) = 0_F\}. \tag{70}$$

Lemma 99 ***(kernel is subspace).*** *Let E and F be spaces. Let $f \in \mathcal{L}(E, F)$ be a linear map from E to F. Then, $\ker(f)$ is a subspace of E.*

Proof. From Lemma 91 (*linear map preserves zero*), and Definition 98 (*kernel*), 0_E belongs to $\ker(f)$. Let $\lambda, \mu \in \mathbb{K}$ be scalars. Let $u, v \in \ker(f)$ be vectors in the kernel. Then, from Lemma 92 (*linear map preserves linear combinations*), Definition 98 (*kernel*), Lemma 72 (*times zero yields zero*), and Definition 58 (*vector space*, $(F, +)$ is an abelian group), we have

$$f(\lambda u + \mu v) = \lambda f(u) + \mu f(v) = \lambda 0_F + \mu 0_F = 0_F.$$

Hence, from Definition 98 (*kernel*), $\lambda u + \mu v$ belongs to $\ker(f)$.

Therefore, from Lemma 79 (*closed under linear combination is subspace*), $\ker(f)$ is a subspace of E. □

Lemma 100 ***(injective linear map has zero kernel).*** *Let E and F be spaces. Let $f \in \mathcal{L}(E,F)$ be a linear map from E to F. Then, f is injective iff $\ker(f) = \{0_E\}$.*

Proof. **"If".** Assume that $\ker(f) = \{0_E\}$. Let $u, v \in E$ be vectors. Assume that $f(u) = f(v)$. Then, from Definition 70 (*vector subtraction*), Definition 62 (*linear map*), and Definition 58 (*vector space*, $(F,+)$ is an abelian group), we have $f(u-v) = 0_F$, hence $u - v$ belongs to $\ker(f)$. Thus $u - v = 0_E$, and from Definition 70 (*vector subtraction*), and Definition 58 (*vector space*, $(E,+)$ is an abelian group), we have $u = v$. Therefore, from **the definition of injectivity**, f is injective.

"Only if". Conversely, assume now that f is injective. Let $u \in \ker(f)$ be a vector in the kernel. Then, from Definition 98 (*kernel*), we have $f(u) = 0_F$. Thus, from Lemma 91 (*linear map preserves zero*), we have $f(u) = f(0_E)$. Therefore, from **the definition of injectivity**, $u = 0_E$. □

Lemma 101 ***(K is space).*** *The commutative field $\mathbb{K}$ equipped with its addition and multiplication is a space.*

Proof. Direct consequence of the commutative field structure. □

4.4 Normed vector space

Definition 102 ***(norm).*** Let E be a space. An application $\|\cdot\| : E \to \mathbb{R}$ is a *norm over E* iff it separates points (or it is definite), it is absolutely homogeneous of degree 1, and it satisfies the triangle inequality:

$$\forall u \in E, \qquad \|u\| = 0 \implies u = 0_E; \tag{71}$$

$$\forall \lambda \in \mathbb{K}, \forall u \in E, \qquad \|\lambda u\| = |\lambda| \, \|u\| ; \tag{72}$$

$$\forall u, v \in E, \qquad \|u + v\| \leq \|u\| + \|v\| . \tag{73}$$

Remark 103. The absolute value over field $\mathbb{K}$ is a function $|\cdot| : \mathbb{K} \to \mathbb{R}$ that is nonnegative, definite, multiplicative, and satisfies the triangle inequality. It is the modulus for the field of complex numbers.

Definition 104 ***(normed vector space).*** $(E, \|\cdot\|)$ is a *normed vector space*, or simply a *normed vector space*, iff E is a space and $\|\cdot\|$ is a norm over E.

Lemma 105 ***(K is normed vector space).*** *The commutative field $\mathbb{K}$ equipped with its absolute value is a normed space.*

Proof. Direct consequence of Definition 104 (*normed vector space*), Definition 102 (*norm*), and **properties of the absolute value over $\mathbb{K}$** (see Remark 103). □

Lemma 106 ***(norm preserves zero).*** *Let $(E, \|\cdot\|)$ be a normed vector space. Then, $\|0_E\| = 0$.*

Proof. From Definition 104 (*normed vector space*, E is a space), and Definition 58 (*vector space*), 0_E belongs to E. From Lemma 68 (*zero times yields zero*), Definition 102 (*norm*, $\|\cdot\|$ is absolutely homogeneous of degree 1), **definition of the absolute value**, and **field properties of $\mathbb{R}$**, we have

$$\|0_E\| = \|0_\mathbb{K} \cdot 0_E\| = |0_\mathbb{K}| \, \|0_E\| = 0_\mathbb{R} \, \|0_E\| = 0_\mathbb{R}.$$

□

Lemma 107 ***(norm is nonnegative).*** *Let $(E, \|\cdot\|)$ be a normed vector space. Then, $\|\cdot\|$ is nonnegative.*

Proof. From Definition 104 (*normed vector space*), E is a space. Let $u \in E$ be a vector. Then, from Definition 102 (*norm*, $\|\cdot\|$ is absolutely homogeneous of degree 1 and satisfies triangle inequality), Definition 58 (*vector space*, $(E,+)$ is an abelian group), and **ordered field properties of $\mathbb{R}$**, we have we $\|-u\| = \|u\|$ and

$$\|u\| = \frac{1}{2}(\|u\| + \|-u\|) \geq \frac{1}{2}\|u-u\| = \frac{1}{2}\|0_E\| = 0.$$

□

Lemma 108 ***(normalization by nonzero).*** *Let $(E, \|\cdot\|)$ be a normed vector space. Then,*

$$\forall \lambda \in \mathbb{K},\, \forall u \in E, \quad u \neq 0 \quad \Longrightarrow \quad \left\| \lambda \frac{u}{\|u\|} \right\| = |\lambda|. \tag{74}$$

Proof. Direct consequence of Definition 71 (*scalar division*), Definition 102 (*norm*, $\|\cdot\|$ is definite and absolutely homogeneous of degree 1), Lemma 107 (*norm is nonnegative*), and **field properties of $\mathbb{R}$**. □

Definition 109 ***(distance associated with norm).*** Let $(E, \|\cdot\|)$ be a normed vector space. The *distance associated with norm* $\|\cdot\|$ is the mapping $d : E \times E \to \mathbb{R}$ defined by

$$\forall u, v \in E, \quad d(u,v) = \|u-v\|. \tag{75}$$

Remark 110. The mapping d will be proved below to be a distance; hence its name.

Lemma 111 ***(norm gives distance).*** *Let $(E, \|\cdot\|)$ be a normed vector space. Let d be the distance associated with norm $\|\cdot\|$. Then, (E,d) is a metric space.*

Proof. From Definition 109 (*distance associated with norm*), Lemma 107 (*norm is nonnegative*), Definition 70 (*vector subtraction*), and Definition 102 (*norm*, $\|\cdot\|$ is absolutely homogeneous of degree 1 with $\lambda = -1$, definite and satisfies triangle inequality), d is nonnegative and symmetric, separates points, and satisfies the triangle inequality. Thus, from Definition 16 (*distance*), d is a distance over E. Therefore, from Definition 17 (*metric space*), (E,d) is a metric space. □

Lemma 112 ***(linear span is closed).*** *Let $(E, \|\cdot\|)$ be a normed vector space. Let d be the distance associated with norm $\|\cdot\|$. Let $u \in E$ be a vector. Then,* $\mathrm{span}(\{u\})$ *is closed for distance d.*

Proof. Let $\lambda, \lambda' \in \mathbb{K}$ be scalars. From Definition 109 (*distance associated with norm*), Definition 58 (*vector space*, scalar multiplication is distributive wrt field addition), Definition 70 (*vector subtraction*), and Definition 102 (*norm*, $\|\cdot\|$ is absolutely homogeneous of degree 1), we have

$$d(\lambda u, \lambda' u) = \|\lambda u - \lambda' u\| = \|(\lambda - \lambda')u\| = |\lambda - \lambda'| \, \|u\|. \tag{76}$$

Case $u = 0_E$. Direct consequence of Definition 80 (*linear span*), and Lemma 24 (*singleton is closed*, $\mathrm{span}(\{u\}) = \{0_E\}$).

Case $u \neq 0_E$. Then, from Definition 102 (*norm*, $\|\cdot\|$ is definite, contrapositive), and Lemma 107 (*norm is nonnegative*), we have $\|u\| > 0$. Let $(\lambda_n u)_{n\in\mathbb{N}}$ be a sequence in $\mathrm{span}(\{u\})$. Assume that this sequence is convergent. Let $\varepsilon > 0$.

From Lemma 36 (*convergent sequence is Cauchy*), **ordered field properties of $\mathbb{R}$ (with $\|u\| > 0$)**, Definition 34 (*Cauchy sequence*, $(\lambda_n u)_{n\in\mathbb{N}}$ is a Cauchy sequence with $\varepsilon \|u\| > 0$), and Equation (76), there exists $N \in \mathbb{N}$ such that for all $p, q \in \mathbb{N}$, $p, q \geq N$ implies

$$|\lambda_p - \lambda_q| = \frac{d(\lambda_p u, \lambda_q u)}{\|u\|} \leq \frac{\varepsilon \|u\|}{\|u\|} = \varepsilon.$$

Hence, from Definition 34 (*Cauchy sequence*), Definition 37 (*complete subset*, $(\lambda_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{K}$ complete), Lemma 28 (*limit is unique*, $(\mathbb{K}, |\cdot|)$ is a metric space), let $\lambda \in \mathbb{K}$ be the limit $\lim_{n\to+\infty} \lambda_n$.

Then, from **ordered field properties of $\mathbb{R}$ (with $\|u\| > 0$)**, Definition 26 (*convergent sequence*, $(\lambda_n)_{n\in\mathbb{N}}$ is convergent with limit λ, with $\frac{\varepsilon}{\|u\|} > 0$), and Equation (76), there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $n \geq N$ implies

$$d(\lambda_n u, \lambda u) = |\lambda_n - \lambda| \, \|u\| \leq \frac{\varepsilon}{\|u\|} \, \|u\| = \varepsilon.$$

Hence, from Definition 26 (*convergent sequence*), $(\lambda_n u)_{n\in\mathbb{N}}$ has limit $\lambda u \in \mathrm{span}(\{u\})$. Therefore, from Lemma 31 (*closed is limit of sequences*), $\mathrm{span}(\{u\})$ is closed (for distance d). □

Definition 113 *(closed unit ball).* Let $(E, \|\cdot\|)$ be a normed vector space. Let d be the distance associated with norm $\|\cdot\|$. The *closed unit ball of E* is $\mathcal{B}_d^c(0_E, 1)$ in the metric space (E, d).

Lemma 114 *(equivalent definition of closed unit ball).* *Let $(E, \|\cdot\|)$ be a normed vector space. Let $\mathcal{B}_1^c$ be the closed unit ball in E. Then, $\mathcal{B}_1^c = \{u \in E \,|\, \|u\| \leq 1\}$.*

Proof. Direct consequence of Definition 113 (*closed unit ball*), Definition 109 (*distance associated with norm*), Lemma 111 (*norm gives distance*), and Definition 19 (*closed ball*). □

Definition 115 *(unit sphere).* Let $(E, \|\cdot\|)$ be a normed vector space. Let d be the distance associated with norm $\|\cdot\|$. The *unit sphere of E* is $\mathcal{S}_d(0_E, 1)$ in the metric space (E, d).

Lemma 116 *(equivalent definition of unit sphere).* *Let $(E, \|\cdot\|)$ be a normed vector space. Let $\mathcal{S}_1$ be the unit sphere in E. Then, $\mathcal{S}_1 = \{u \in E \,|\, \|u\| = 1\}$.*

Proof. Direct consequence of Definition 115 (*unit sphere*), Definition 109 (*distance associated with norm*), Lemma 111 (*norm gives distance*), and Definition 20 (*sphere*). □

Lemma 117 *(zero on unit sphere is zero).* *Let $(E, \|\cdot\|)$ be a normed vector space. Let $\mathcal{S}_1$ be the unit sphere in E. Let F be a space. Let $f \in \mathcal{L}(E, F)$ be a linear map from E to F. Then, $f = 0_{\mathcal{L}(E,F)}$ iff f is zero on $\mathcal{S}_1$.*

Proof. **"If".** Assume that f is zero on the unit sphere. Let $u \in E$ be a vector.

Case $u = 0_E$. Then, from Lemma 91 (*linear map preserves zero*), $f(u) = f(0_E) = 0_F$.

Case $u \neq 0_E$. Then, from Lemma 108 (*normalization by nonzero*, with $\lambda = 1$), and Lemma 116 (*equivalent definition of unit sphere*), $\xi = \frac{u}{\|u\|}$ belongs to $\mathcal{S}_1$. Thus, from Definition 58 (*vector space*, scalar multiplication is compatible with field multiplication), and **field properties of $\mathbb{R}$**, we have $u = \|u\| \, \xi$. Hence, from Definition 62 (*linear map*, homogeneity of degree 1), hypothesis, and Lemma 72 (*times zero yields zero*), we have

$$f(u) = f(\|u\| \, \xi) = \|u\| \, f(\xi) = \|u\| \, 0_F = 0_F.$$

Therefore, in both cases, $f = 0_{\mathcal{L}(E,F)}$.

"Only if". Conversely, assume now that $f = 0_{\mathcal{L}(E,F)}$. Then, from Lemma 116 (*equivalent definition of unit sphere*, $\mathcal{S}_1$ is a subset of E), f is also zero on the unit sphere. □

Lemma 118 *(reverse triangle inequality).* *Let $(E, \|\cdot\|)$ be a normed vector space. Then,*

$$\forall u, v \in E, \quad |\|u\| - \|v\|| \leq \|u - v\| \,. \tag{77}$$

Proof. Let $u, v \in E$ be vectors. Then, from Definition 104 (*normed vector space*, $\|\cdot\|$ is a norm), and Definition 102 (*norm*, $\|\cdot\|$ satisfies triangle inequality), we have $\|u\| \leq \|u - v\| + \|v\|$. Hence, from **ordered field properties of $\mathbb{R}$**, we have $\|u\| - \|v\| \leq \|u - v\|$. Thus, from Definition 102 (*norm*, $\|\cdot\|$ is absolutely homogeneous of degree 1 with $\lambda = -1$), we have

$$\|v\| - \|u\| \leq \|v - u\| = \|u - v\| .$$

Therefore, from **properties of the absolute value in $\mathbb{R}$**, we have $|\|u\| - \|v\|| \leq \|u - v\|$. □

Lemma 119 ***(norm is one-Lipschitz continuous).*** *Let $(E, \|\cdot\|)$ be a normed vector space. Let d be the distance associated with norm $\|\cdot\|$. Then, $\|\cdot\|$ is 1-Lipschitz continuous from (E, d) to $(\mathbb{R}, |\cdot|)$.*

Proof. Direct consequence of Lemma 118 (*reverse triangle inequality*), Definition 109 (*distance associated with norm*), and Definition 46 (*Lipschitz continuity*, with $k = 1$). □

Lemma 120 ***(norm is uniformly continuous).*** *Let $(E, \|\cdot\|)$ be a normed vector space. Let d be the distance associated with norm $\|\cdot\|$. Then, $\|\cdot\|$ is uniformly continuous from (E, d) to $(\mathbb{R}, |\cdot|)$.*

Proof. Direct consequence of Lemma 119 (*norm is one-Lipschitz continuous*), and Definition 51 (*Lipschitz continuous is uniform continuous*). □

Lemma 121 ***(norm is continuous).*** *Let $(E, \|\cdot\|)$ be a normed vector space. Let d be the distance associated with norm $\|\cdot\|$. Then, $\|\cdot\|$ is continuous from (E, d) to $(\mathbb{R}, |\cdot|)$.*

Proof. Direct consequence of Lemma 120 (*norm is uniformly continuous*), and Lemma 49 (*uniform continuous is continuous*). □

Definition 122 ***(linear isometry).*** Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be normed vector spaces. Let $f \in \mathcal{L}(E, F)$ be a linear map from E to F. f is a *linear isometry from E to F* iff it preserves the norm:

$$\forall u \in E, \quad \|f(u)\|_F = \|u\|_E . \tag{78}$$

Lemma 123 ***(identity map is linear isometry).*** *Let $(E, \|\cdot\|)$ be a normed vector space. Then, the identity map Id_E is a linear isometry.*

Proof. Direct consequence of Lemma 95 (*identity map is linear map*), Definition 94 (*identity map*), and Definition 122 (*linear isometry*). □

Definition 124 ***(product norm).*** Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be normed vector spaces. The *product norm induced over $E \times F$* is the mapping $\|(\cdot, \cdot)\|_{E \times F} : E \times F \to \mathbb{R}$ defined by

$$\forall (u, v) \in E \times F, \quad \|(u, v)\|_{E \times F} = \|u\|_E + \|v\|_F . \tag{79}$$

Remark 125. The mapping $\|(\cdot, \cdot)\|_{E \times F}$ will be proved below to be a norm; hence its name and notation.

Remark 126. The norm $\|(\cdot, \cdot)\|_{E \times F}$ is the L^1-like norm over the product $E \times F$. L^p-like norms for $p \geq 1$ and $p = +\infty$ are also possible; they are all equivalent norms.

Lemma 127 ***(product is normed vector space).*** *Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be normed vector spaces. Let $\|(\cdot, \cdot)\|_{E \times F}$ be the product norm induced over $E \times F$. Then, $(E \times F, \|(\cdot, \cdot)\|_{E \times F})$ is a normed vector space.*

Proof. From Lemma 87 (*product is space*), $E \times F$, equipped with product vector operations of Definition 86 (*product vector operations*), is a space.

Let $(u, v) \in E \times F$ be vectors. Assume that $\|(u, v)\|_{E\times F} = 0$. Then, from Definition 124 (*product norm*), we have $\|u\|_E + \|v\|_F = 0$. And, from Lemma 107 (*norm is nonnegative*, for $\|\cdot\|_E$ and $\|\cdot\|_F$), and **ordered field properties of** $\mathbb{R}$, we have $\|u\|_E = \|v\|_F = 0$. Thus, from Definition 102 (*norm*, $\|\cdot\|_E$ and $\|\cdot\|_F$ are definite), and Lemma 87 (*product is space*, $0_{E\times F} = (0_E, 0_F)$), we have $(u, v) = 0_{E\times F}$. Hence, $\|(\cdot,\cdot)\|_{E\times F}$ is definite.

Let $\lambda \in \mathbb{K}$ be a scalar. Let $(u, v) \in E \times F$ be vectors. Then, from Definition 86 (*product vector operations*, scalar multiplication), Definition 124 (*product norm*), Definition 102 (*norm*, $\|\cdot\|_E$ and $\|\cdot\|_F$ are absolutely homogeneous of degree 1), and **field properties of** $\mathbb{R}$, we have

$$\begin{aligned}\|\lambda(u, v)\|_{E\times F} &= \|(\lambda u, \lambda v)\|_{E\times F} = \|\lambda u\|_E + \|\lambda v\|_F \\ &= |\lambda|\ \|u\|_E + |\lambda|\ \|v\|_F = |\lambda|(\|u\|_E + \|v\|_F) = |\lambda|\ \|(u, v)\|_{E\times F}\,.\end{aligned}$$

Hence, $\|(\cdot,\cdot)\|_{E\times F}$ is absolutely homogeneous of degree 1.

Let $(u, v), (u', v') \in E \times F$ be vectors. Then, from Definition 86 (*product vector operations*, vector addition), Definition 124 (*product norm*), Definition 102 (*norm*, $\|\cdot\|_E$ and $\|\cdot\|_F$ satisfy triangle inequality), and **field properties of** $\mathbb{R}$, we have

$$\begin{aligned}\|(u, v) + (u', v')\|_{E\times F} &= \|(u + u', v + v')\|_{E\times F} = \|u + u'\|_E + \|v + v'\|_F \\ &\leq (\|u\|_E + \|u'\|_E) + (\|v\|_F + \|v'\|_F) = (\|u\|_E + \|v\|_F) + (\|u'\|_E + \|v'\|_F) \\ &= \|(u, v)\|_{E\times F} + \|(u', v')\|_{E\times F}\,.\end{aligned}$$

Hence, $\|(\cdot,\cdot)\|_{E\times F}$ satisfies triangle inequality.

Therefore, from Definition 102 (*norm*), $\|(\cdot,\cdot)\|_{E\times F}$ is a norm over $E \times F$, hence, from Definition 104 (*normed vector space*), $(E \times F, \|(\cdot,\cdot)\|_{E\times F})$ is a normed vector space. □

Lemma 128 *(vector addition is continuous).* *Let $(E, \|\cdot\|_E)$ be a normed vector space. From Lemma 127 (*product is normed vector space*), let $\|(\cdot,\cdot)\|_{E\times E}$ be the product norm induced over $E \times E$. Let d_E and $d_{E\times E}$ be the distances associated with norms $\|\cdot\|_E$ and $\|(\cdot,\cdot)\|_{E\times E}$. Then, the vector addition is continuous from $(E \times E, d_{E\times E})$ to (E, d_E).*

Proof. Let $u, v, u', v' \in E$. From Definition 109 (*distance associated with norm*), Definition 58 (*vector space*, $(E, +)$ is an abelian group), Definition 102 (*norm*, $\|\cdot\|_E$ satisfies triangle inequality), Definition 124 (*product norm*), and Definition 86 (*product vector operations*), we have

$$\begin{aligned}d_E(u + v, u' + v') &= \|(u + v) - (u' + v')\|_E = \|u - u' + v - v'\|_E \\ &\leq \|u - u'\|_E + \|v - v'\|_E = \|(u - u', v - v')\|_{E\times E} = \|(u, v) - (u', v')\|_{E\times E} \\ &= d_{E\times E}((u, v), (u', v')).\end{aligned}$$

Let $u, v \in E$ and $\varepsilon > 0$. Set $\delta = \varepsilon$, then for all $u', v' \in E$, we have

$$d_{E\times E}((u, v), (u', v')) \leq \delta \quad\Longrightarrow\quad d_E(u + v, u' + v') \leq \delta = \varepsilon.$$

Therefore, from Definition 42 (*continuity in a point*), and Definition 43 (*pointwise continuity*), the vector addition is (pointwise) continuous from $(E \times E, d_{E\times E})$ to (E, d_E). □

Lemma 129 *(scalar multiplication is continuous).* *Let $(E, \|\cdot\|)$ be a normed vector space. Let d be the distance associated with norm $\|\cdot\|$. Let $\lambda \in \mathbb{K}$ be a scalar. Then, the scalar multiplication by λ is continuous from (E, d) to itself.*

Proof. **Case $\boldsymbol{\lambda = 0_{\mathbb{K}}}$.** Let $u \in E$ and $\varepsilon > 0$. Set $\delta = 1$. Then, for all $u' \in E$, from Lemma 68 (*zero times yields zero*), and Definition 16 (*distance*, d separates points), we have

$$d(u, u') \leq \delta = 1 \quad\Longrightarrow\quad d(\lambda u, \lambda u') = d(0_E, 0_E) = 0 \leq \varepsilon.$$

Therefore, from Definition 42 (*continuity in a point*), and Definition 43 (*pointwise continuity*), the scalar multiplication by 0_E is (pointwise) continuous from (E, d) to itself.

Case $\boldsymbol{\lambda \neq 0_{\mathbb{K}}}$. Let $u, u' \in E$. From Definition 109 (*distance associated with norm*), Definition 58 (*vector space*, scalar multiplication is distributive wrt vector addition), and Definition 102 (*norm*, $\|\cdot\|$ is absolutely homogeneous of degree 1), we have

$$d(\lambda u, \lambda u') = \|\lambda u - \lambda u'\| = |\lambda| \, \|u - u'\| = |\lambda| d(u, u').$$

Let $u \in E$ and $\varepsilon > 0$. From **properties of the absolute value over $\mathbb{K}$**, $|\lambda| \neq 0$ and we can set $\delta = \frac{\varepsilon}{|\lambda|}$. Then for all $u' \in E$, we have

$$d(u, u') \leq \delta \quad \Longrightarrow \quad d(\lambda u, \lambda u') \leq |\lambda| \delta = \varepsilon.$$

Therefore, from Definition 42 (*continuity in a point*), and Definition 43 (*pointwise continuity*), the scalar multiplication is (pointwise) continuous from (E, d) to itself. □

4.4.1 Topology

Remark 130. Since a distance can be defined from a norm, normed vector spaces can be seen as metric spaces, hence as topological spaces too. Therefore, the important notions of *continuous* linear map and of *closed* subspace.

Remark 131. There exists a purely algebraic notion of dual of a space E: the space of linear forms over E, usually denoted $E^\star = \mathcal{L}(E, \mathbb{K})$. We focus here on the notion of *topological* dual of a normed vector space E: the space of *continuous* linear forms over E, usually denoted $E' = \mathcal{L}_c(E, \mathbb{K})$.

Remark 132. When W is a subset of the set X, and f a mapping from X to Y, the notation $f(W)$ denotes the subset of Y made of the images of elements of W. Applied to a norm on a vector space, when X is a subset of a normed space $(E, \|\cdot\|)$, the notation $\|X\|$ denotes the subset of $\mathbb{R}$ of values taken by norm $\|\cdot\|$ on vectors of X:

$$\|X\| = \{\|u\| \mid u \in X\}.$$

4.4.1.1 Continuous linear map

Lemma 133 *(norm of image of unit vector).* *Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be normed vector spaces. Let $\mathcal{S}_1$ be the unit sphere in E. Let $f \in \mathcal{L}(E, F)$ be a linear map from E to F. Let $u \in E$ be a vector. Assume that $u \neq 0_E$. Then, $\frac{u}{\|u\|_E}$ belongs to $\mathcal{S}_1$ and*

$$\left\| f\left(\frac{u}{\|u\|_E}\right)\right\|_F = \frac{\|f(u)\|_F}{\|u\|_E}. \tag{80}$$

Proof. From Definition 102 (*norm*, $\|\cdot\|_E$ is definite, contrapositive), we have $\|u\|_E \neq 0$. Thus, from Lemma 107 (*norm is nonnegative*), and **field properties of $\mathbb{R}$**, $\frac{1}{\|u\|_E} \geq 0$. Let $\xi = \frac{u}{\|u\|_E}$. Then, from Lemma 108 (*normalization by nonzero*, with $\lambda = 1$), we have $\|\xi\|_E = 1$. Hence, ξ belongs to $\mathcal{S}_1$.

From Definition 62 (*linear map*, homogeneity of degree 1), Definition 102 (*norm*, $\|\cdot\|_F$ is absolutely homogeneous of degree 1), and Lemma 107 (*norm is nonnegative*), we have

$$\|f(\xi)\|_F = \left\| f\left(\frac{u}{\|u\|_E}\right)\right\|_F = \left\| \frac{f(u)}{\|u\|_E}\right\|_F = \frac{\|f(u)\|_F}{\|u\|_E}.$$

□

Lemma 134 ***(norm of image of unit sphere).*** *Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be normed vector spaces. Let $\mathcal{S}_1$ be the unit sphere in E. Let $f \in \mathcal{L}(E, F)$ be a linear map from E to F. Then,*

$$\|f(\mathcal{S}_1)\|_F = \left\{ \frac{\|f(u)\|_F}{\|u\|_E} \;\middle|\; u \in E,\, u \neq 0_E \right\}. \tag{81}$$

Proof. From Definition 102 (*norm*, $\|\cdot\|_E$ is definite, contrapositive), and **field properties of $\mathbb{R}$**, let $g : E \to \mathbb{R}$ be the mapping defined by $g(0_E) = 0$, and for all $u \in E\backslash\{0_E\}$, $g(u) = \frac{\|f(u)\|_F}{\|u\|_E}$.

Let $\xi \in \mathcal{S}_1$ be a unit vector. From Lemma 116 (*equivalent definition of unit sphere*), $\|\xi\|_E = 1$. Then, from Lemma 106 (*norm preserves zero*, contrapositive), $\xi \neq 0_E$. Thus, from **field properties of $\mathbb{R}$**, we have

$$\|f(\xi)\|_F = \frac{\|f(\xi)\|_F}{\|\xi\|_E} = g(\xi) \in g(E\backslash\{0_E\}).$$

Hence, $\|f(\mathcal{S}_1)\|_F \subset g(E\backslash\{0_E\})$.

Let $u \in E$ be a vector. Assume that $u \neq 0_E$. Then, from Lemma 133 (*norm of image of unit vector*), $\xi = \frac{u}{\|u\|_E}$ belongs to $\mathcal{S}_1$ and

$$g(u) = \frac{\|f(u)\|_F}{\|u\|_E} = \|f(\xi)\|_F \in f(\mathcal{S}_1).$$

Hence, $g(E\backslash\{0_E\}) \subset \|f(\mathcal{S}_1)\|_F$.

Therefore, $\|f(\mathcal{S}_1)\|_F = g(E\backslash\{0_E\})$. □

Definition 135 ***(operator norm).*** Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be normed vector spaces. Let $f \in \mathcal{L}(E, F)$ be a linear map from E to F. The *operator norm on $\mathcal{L}(E, F)$ induced by norms on E and F* is the mapping $N_{E,F} : \mathcal{L}(E, F) \to \overline{\mathbb{R}}$ defined by

$$N_{E,F}(f) = \sup \left\{ \frac{\|f(u)\|_F}{\|u\|_E} \;\middle|\; u \in E,\, u \neq 0_E \right\}. \tag{82}$$

Remark 136. When restricted to continuous linear maps, the mapping $N_{E,F}$ will be proved below to be a norm; hence its name.

Lemma 137 ***(equivalent definition of operator norm).*** *Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be normed vector spaces. Let $\mathcal{S}_1$ be the unit sphere in E. Let $f \in \mathcal{L}(E, F)$ be a linear map from E to F. Then,*

$$N_{E,F}(f) = \sup(\|f(\mathcal{S}_1)\|_F). \tag{83}$$

Proof. Direct consequence of Definition 135 (*operator norm*), and Lemma 134 (*norm of image of unit sphere*). □

Lemma 138 ***(operator norm is nonnegative).*** *Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be normed vector spaces. Then, $N_{E,F}$ is nonnegative.*

Proof. Let $f \in \mathcal{L}(E, F)$ be a linear map from E to F. Then, from Lemma 137 (*equivalent definition of operator norm*), we have $N_{E,F}(f) = \sup(\|f(\mathcal{S}_1)\|_F)$. Let $\xi \in \mathcal{S}_1$ be a unit vector. Then, from Lemma 107 (*norm is nonnegative*), $\|f(\xi)\|_F$ is nonnegative. Therefore, from Definition 2 (*supremum*, $N_{E,F}(f)$ is an upper bound for $\|f(\mathcal{S}_1)\|_F$), $N_{E,F}(f)$ is nonnegative too. □

Definition 139 ***(bounded linear map).*** Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be normed vector spaces. A linear map f from E to F is *bounded* iff

$$\exists C \geq 0,\, \forall u \in E, \quad \|f(u)\|_F \leq C\, \|u\|_E\,. \tag{84}$$

Then, C is called *continuity constant of f*.

Definition 140 ***(linear map bounded on unit ball).*** Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be normed vector spaces. Let $\mathcal{B}_1^c$ be the closed unit ball in E. A linear map f from E to F is *bounded on the closed unit ball* iff there exists an upper bound for $\|f(\mathcal{B}_1^c)\|_F$, i.e.

$$\exists C \geq 0, \forall \xi \in \mathcal{B}_1^c, \quad \|f(\xi)\|_F \leq C. \tag{85}$$

Definition 141 ***(linear map bounded on unit sphere).*** Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be normed vector spaces. Let $\mathcal{S}_1$ be the unit sphere in E. A linear map f from E to F is *bounded on the unit sphere* iff there exists an upper bound for $\|f(\mathcal{S}_1)\|_F$, i.e.

$$\exists C \geq 0, \forall \xi \in \mathcal{S}_1, \quad \|f(\xi)\|_F \leq C. \tag{86}$$

Theorem 142 ***(continuous linear map).*** *Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be normed vector spaces. Let $f \in \mathcal{L}(E, F)$ be a linear map from E to F. Then, the following propositions are equivalent:*

1. *f is continuous in 0_E;*
2. *f is continuous;*
3. *f is uniformly continuous;*
4. *f is Lipschitz continuous;*
5. *f is bounded;*
6. *$N_{E,F}(f)$ is finite;*
7. *f is bounded on the unit sphere.*
8. *f is bounded on the closed unit ball.*

Proof. Let $\mathcal{S}_1$ be the unit sphere in E. Let $\mathcal{B}_1^c$ be the closed unit ball in E.

5 implies 4. Assume that f is bounded. From Definition 139 (*bounded linear map*), let $C \geq 0$ such that, for all $u \in E$, we have $\|f(u)\|_F \leq C \|u\|_E$. Let $u, v \in E$ be vectors. Then, from Definition 70 (*vector subtraction*), Definition 62 (*linear map*), and hypothesis, we have

$$\|f(u) - f(v)\|_F = \|f(u - v)\|_F \leq C \|u - v\|_E.$$

Hence, from Definition 46 (*Lipschitz continuity*), f is C-Lipschitz continuous.

4 implies 3. Assume that f is Lipschitz continuous. Then, from Lemma 51 (*Lipschitz continuous is uniform continuous*), f is uniformly continuous.

3 implies 2. Assume that f is uniformly continuous. Then, from Lemma 49 (*uniform continuous is continuous*), f is (pointwise) continuous.

2 implies 1. Assume that f is (pointwise) continuous. Then, from Definition 43 (*pointwise continuity*), f is continuous in 0_E.

1 implies 8. Assume now that f is continuous in 0_E. Let $\varepsilon = 1 > 0$. Then, from Definition 42 (*continuity in a point*), and Lemma 91 (*linear map preserves zero*), let $\delta > 0$ such that, for all $u \in E$, $\|u - 0_E\|_E = \|u\|_E \leq \delta$ implies $\|f(u) - f(0_E)\|_F = \|f(u)\|_F \leq 1$. Let $C = \frac{1}{\delta} > 0 \geq 0$. Let $\xi \in \mathcal{B}_1^c$ be a vector in the unit ball. From Lemma 114 (*equivalent definition of closed unit ball*), $\|\xi\|_E \leq 1$. Then, from Definition 102 (*norm*, $\|\cdot\|_E$ is absolutely homogeneous of degree 1), and **ordered field properties of $\mathbb{R}$**, we have $\|\delta\xi\|_E \leq \delta \|\xi\|_E \leq \delta$. Thus, from Definition 102 (*norm*, $\|\cdot\|_F$ is absolutely homogeneous of degree 1), Definition 62 (*linear map*, homogeneity of degree 1), **ordered field properties of $\mathbb{R}$**, and hypothesis, we have $\|f(\xi)\|_F = \frac{1}{\delta} \|f(\delta\xi)\|_F \leq \frac{1}{\delta} = C$. Hence, from Definition 140 (*linear map bounded on unit ball*), f is bounded on the unit ball.

8 implies 7. Assume now that f is bounded on the unit ball. From Definition 140 (*linear map bounded on unit ball*), and Lemma 114 (*equivalent definition of closed unit ball*), let $C \geq 0$ such that for all $\xi \in \mathcal{B}_1^c$, $\|f(\xi)\|_F \leq C$. Let $\xi \in \mathcal{S}_1$ be a unit vector. Then, from Lemma 116 (*equivalent definition of unit sphere*), and Lemma 114 (*equivalent definition of closed unit ball*), we also have $\xi \in \mathcal{B}_1^c$. Thus, from hypothesis, $\|f(\xi)\|_F \leq C$. Hence, from Definition 141 (*linear map bounded on unit sphere*, with same constant C), f is bounded on the unit sphere.

7 implies 6. Assume then that f is bounded on the unit sphere. Then, from Definition 141 (*linear map bounded on unit sphere*), there exists a finite upper bound $C \geq 0$ for $\|f(\mathcal{S}_1)\|_F$. Hence, from Lemma 3 (*finite supremum*), $\sup(\|f(\mathcal{S}_1)\|_F)$ is finite, and from Lemma 137 (*equivalent definition of operator norm*), $N_{E,F}(f)$ is finite.

6 implies 5. Assume finally that $N_{E,F}(f)$ is finite. Let $C = N_{E,F}(f)$. Then, from Lemma 138 (*operator norm is nonnegative*), C is nonnegative. Let $u \in E$ be a vector.

Case $u = 0_E$. Then, from Lemma 91 (*linear map preserves zero*), $f(u) = f(0_E) = 0_F$. Hence, from Lemma 106 (*norm preserves zero*, for $\|\cdot\|_F$ and $\|\cdot\|_E$), and **ordered field properties of $\mathbb{R}$**, we have

$$\|f(u)\|_F = 0 \leq 0 = C\,0 = C\,\|u\|_E\,.$$

Case $u \neq 0_E$. Then, from Definition102 (*norm*, $\|\cdot\|_E$ is definite, contrapositive), $\|u\|_E \neq 0$. Thus, from **field properties of $\mathbb{R}$**, Definition 135 (*operator norm*), and Definition 2 (*supremum*, $N_{E,F}(f)$ is an upper bound for $\left\{ \frac{\|f(u)\|_F}{\|u\|_E} \,\middle|\, u \in E,\, u \neq 0_E \right\}$), we have

$$\|f(u)\|_F = \frac{\|f(u)\|_F}{\|u\|_E}\,\|u\|_E \leq N_{E,F}(f)\,\|u\|_E = C\,\|u\|_E\,.$$

Hence, from Definition 139 (*bounded linear map*), f is bounded.

Therefore, we have $5 \Rightarrow 4 \Rightarrow 3 \Rightarrow 2 \Rightarrow 1 \Rightarrow 8 \Rightarrow 7 \Rightarrow 6 \Rightarrow 5$, hence all properties are equivalent. □

Definition 143 *(set of continuous linear maps).* Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be normed vector spaces. The *set of continuous linear maps from E to F* is denoted $\mathcal{L}_c(E,F)$.

Lemma 144 *(finite operator norm is continuous).* *Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be normed vector spaces. Let $f \in \mathcal{L}(E,F)$ be a linear map from E to F. Then, f belongs to $\mathcal{L}_c(E,F)$ (i.e. f is continuous) iff $N_{E,F}(f)$ is finite. Moreover, let $\mathcal{B}_1^c$ and $\mathcal{S}_1$ be the closed unit ball and the unit sphere in E, and let $C \geq 0$, then we have the following equivalences:*

$$\begin{aligned} N_{E,F}(f) \leq C \quad &\Leftrightarrow \quad C \text{ is an upper bound for } \left\{ \frac{\|f(u)\|_F}{\|u\|_E} \,\middle|\, u \in E,\, u \neq 0_E \right\} \\ &\Leftrightarrow \quad C \text{ is a continuity constant for } f \\ &\Leftrightarrow \quad C \text{ is an upper bound for } \|f(\mathcal{B}_1^c)\|_F \\ &\Leftrightarrow \quad C \text{ is an upper bound for } \|f(\mathcal{S}_1)\|_F\,. \end{aligned} \tag{87}$$

Proof. Direct consequences of Definition 143 (*set of continuous linear maps*), Theorem 142 (*continuous linear map*, $2 \Rightarrow 6$). Definition 2 (*supremum*, $N_{E,F}(f)$ is the least upper bound of $\left\{ \frac{\|f(u)\|_F}{\|u\|_E} \,\middle|\, u \in E,\, u \neq 0_E \right\}$), Definition 139 (*bounded linear map*), Definition 140 (*linear map bounded on unit ball*), and Definition 141 (*linear map bounded on unit sphere*). □

Lemma 145 *(linear isometry is continuous).* *Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be normed vector spaces. Let $f \in \mathcal{L}(E,F)$ be a linear map from E to F. Assume that f is a linear isometry from E to F. Then, f belongs to $\mathcal{L}_c(E,F)$ (i.e. f is continuous).*

Proof. Let $\mathcal{S}_1$ be the unit sphere in E. Let $\xi \in \mathcal{S}_1$ be a unit vector. Then, from Definition 122 (*linear isometry*), we have $\|f(\xi)\|_F = \|\xi\|_E = 1 \leq 1$. Hence, from Lemma 144 (*finite operator norm is continuous*, 1 is an upper bound for $\|f(\mathcal{S}_1)\|_F$), f belongs to $\mathcal{L}_c(E, F)$. □

Lemma 146 ***(identity map is continuous).*** *Let $(E, \|\cdot\|_E)$ be a normed vector space. Then, the identity map Id_E belongs to $\mathcal{L}_c(E, E)$ (i.e. Id_E is continuous).*

Proof. Direct consequence of Lemma 123 (*identity map is linear isometry*), and Lemma 145 (*linear isometry is continuous*). □

Theorem 147 ***(normed vector space of continuous linear maps).*** *Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be normed vector spaces. Let $|||\cdot|||_{F,E}$ be the restriction of $N_{E,F}$ to continuous linear maps. Then, $(\mathcal{L}_c(E, F), |||\cdot|||_{F,E})$ is a normed vector space.*

Proof. Let $\mathcal{S}_1$ be the unit sphere in E. From Definition 143 (*set of continuous linear maps*), $\mathcal{L}_c(E, F)$ is obviously a subset of $\mathcal{L}(E, F)$.

Let $f \in \mathcal{L}_c(E, F)$ be a continuous linear map from E to F. Then, from Lemma 144 (*finite operator norm is continuous*), $|||f|||_{F,E}$ is finite. Hence, $|||\cdot|||_{F,E}$ is a mapping from $\mathcal{L}_c(E, F)$ to $\mathbb{R}$.

From Definition 141 (*linear map bounded on unit sphere*, with $C = 0$), and Lemma 144 (*finite operator norm is continuous*, upper bound for $\|0_{\mathcal{L}(E,F)}(\mathcal{S}_1)\|_F$), $0_{\mathcal{L}(E,F)}$ belongs to $\mathcal{L}_c(E, F)$.

Let $f \in \mathcal{L}_c(E, F)$ be a continuous linear map from E to F. Assume that $|||f|||_{F,E} = 0$. Let $\xi \in \mathcal{S}_1$ be a unit vector. Then, from Lemma 107 (*norm is nonnegative*), Lemma 137 (*equivalent definition of operator norm*), and Definition 2 (*supremum*, $|||f|||_{F,E}$ is an upper bound for $\|f(\mathcal{S}_1)\|_F$), we have

$$0 \leq \|f(\xi)\|_F \leq |||f|||_{F,E} = 0.$$

Thus, $\|f(\xi)\|_F = 0$, and from Definition 102 (*norm*, $\|\cdot\|_F$ is definite), $f(\xi) = 0_F$. Hence, from Lemma 117 (*zero on unit sphere is zero*), $f = 0_{\mathcal{L}(E,F)}$, and $|||\cdot|||_{F,E}$ is definite.

Let $\lambda \in \mathbb{K}$ be a scalar. Let $f \in \mathcal{L}_c(E, F)$ be a continuous linear map from E to F. Let $\xi \in \mathcal{S}_1$ be a unit vector. Then, from Definition 88 (*inherited vector operations*, scalar multiplication), and Definition 102 (*norm*, $\|\cdot\|_F$ is absolutely homogeneous of degree 1), we have

$$\|(\lambda f)(\xi)\|_F = \|\lambda f(\xi)\|_F = |\lambda| \, \|f(\xi)\|_F .$$

Thus, from Lemma 137 (*equivalent definition of operator norm*), Lemma 5 (*supremum is positive scalar multiplicative*), and nonnegativeness of absolute value, we have

$$N_{E,F}(\lambda f) = \sup(\|(\lambda f)(\mathcal{S}_1)\|_F) = |\lambda| \sup(\|f(\mathcal{S}_1)\|_F) = |\lambda| \, |||f|||_{F,E}.$$

Then, from Lemma 144 (*finite operator norm is continuous*, $N_{E,F}(\lambda f)$ is finite), λf belongs to $\mathcal{L}_c(E, F)$. Hence, $|||\cdot|||_{F,E}$ is absolutely homogeneous of degree 1, and $\mathcal{L}_c(E, F)$ is closed under scalar multiplication.

Let $f, g \in \mathcal{L}_c(E, F)$ be continuous linear maps from E to F. Let $\xi \in \mathcal{S}_1$ be a unit vector. Then, from Definition 88 (*inherited vector operations*, vector addition), Definition 102 (*norm*, $\|\cdot\|_F$ satisfies triangle inequality), Lemma 137 (*equivalent definition of operator norm*), Definition 2 (*supremum*, $|||f|||_{F,E}$, resp. $|||g|||_{F,E}$, is an upper bounds for $\|f(\mathcal{S}_1)\|_F$, resp. $\|g(\mathcal{S}_1)\|_F$), and **field properties of $\mathbb{R}$**, we have

$$\|(f+g)(\xi)\|_F = \|f(\xi) + g(\xi)\|_F \leq \|f(\xi)\|_F + \|g(\xi)\|_F \leq |||f|||_{F,E} + |||g|||_{F,E}.$$

Thus, from Lemma 144 (*finite operator norm is continuous*, $|||f|||_{F,E} + |||g|||_{F,E}$ is a finite upper bound for $\|(f+g)(\mathcal{S}_1)\|_F$), $f + g$ belongs to $\mathcal{L}_c(E, F)$ and

$$|||f+g|||_{F,E} \leq |||f|||_{F,E} + |||g|||_{F,E}.$$

Hence, $\mathcal{L}_c(E, F)$ is closed under vector addition and $|||\cdot|||_{F,E}$ satisfies triangle inequality.

Therefore, from Lemma 78 (*closed under vector operations is subspace*), Definition 102 (*norm*), and Definition 104 (*normed vector space*), $\mathcal{L}_c(E, F)$ is a subspace of $\mathcal{L}(E, F)$, $|||\cdot|||_{F,E}$ is a norm over $\mathcal{L}(E, F)$, and $(\mathcal{L}_c(E, F), |||\cdot|||_{F,E})$ is a normed vector space. □

Lemma 148 ***(operator norm estimation).*** *Let* $(E, \|\cdot\|_E)$ *and* $(F, \|\cdot\|_F)$ *be normed vector spaces. Then,*

$$\forall f \in \mathcal{L}_c(E,F), \forall u \in E, \quad \|f(u)\|_F \leq |||f|||_{F,E} \, \|u\|_E\,. \tag{88}$$

Proof. Let $f \in \mathcal{L}_c(E,F)$ be a continuous linear map from E to F. Then, from Theorem 147 (*normed vector space of continuous linear maps*), $|||f|||_{F,E}$ is finite. Let $u \in E$ be a vector.

Case $u = 0_E$. Then, from Lemma 91 (*linear map preserves zero*), Lemma 106 (*norm preserves zero*, for $\|\cdot\|_F$ and $\|\cdot\|_E$), and **ordered field properties of $\mathbb{R}$**, we have

$$\|f(u)\|_F = \|f(0_E)\|_F = \|0_F\|_F = 0 \leq 0 = |||f|||_{F,E}\, 0 = |||f|||_{F,E}\, \|0_E\|_E = |||f|||_{F,E}\, \|u\|_E\,.$$

Case $u \neq 0_E$. Then, from Definition 135 (*operator norm*), and Definition 2 (*supremum*, $|||f|||_{F,E}$ is an upper bound for $\left\{ \frac{\|f(u)\|_F}{\|u\|_E} \;\middle|\; u \in E,\ u \neq 0_E \right\}$), we have $\frac{\|f(u)\|_F}{\|u\|_E} \leq |||f|||_{F,E}$. From Definition 102 (*norm*, $\|\cdot\|_E$ is definite, contrapositive), and Lemma 107 (*norm is nonnegative*, for $\|\cdot\|_E$), $\|u\|_E > 0$. Hence, from **ordered field properties of $\mathbb{R}$**, $\|f(u)\|_F \leq |||f|||_{F,E}\, \|u\|_E$. □

Lemma 149 ***(continuous linear maps have closed kernel).*** *Let* $(E, \|\cdot\|_E)$ *and* $(F, \|\cdot\|_F)$ *be normed vector spaces. Let* $f \in \mathcal{L}_c(E,F)$ *be a continuous linear map from* E *to* F*. Then,* $\ker(f)$ *is closed in* E*.*

Proof. Direct consequence of Definition 98 (*kernel*, $\ker(f) = f^{-1}\{0_F\}$), Lemma 24 (*singleton is closed*, $\{0_F\}$ is closed), and **preimages of closed subsets by continuous mappings are closed**. □

Lemma 150 ***(compatibility of composition with continuity).*** *Let* $(E, \|\cdot\|_E)$*,* $(F, \|\cdot\|_F)$ *and* $(G, \|\cdot\|_G)$ *be normed vector spaces. Then,*

$$\forall f \in \mathcal{L}_c(E,F), \forall g \in \mathcal{L}_c(F,G), \quad g \circ f \in \mathcal{L}_c(E,G) \quad \wedge \quad |||g \circ f|||_{G,E} \leq |||g|||_{G,F}\, |||f|||_{F,E}. \tag{89}$$

Proof. Let $\mathcal{S}_1$ be the unit sphere in E. Let $f \in \mathcal{L}_c(E,F)$ and $g \in \mathcal{L}_c(F,G)$ be continuous linear maps. Then, from Lemma 96 (*composition of linear maps is bilinear*), $g \circ f$ belongs to $\mathcal{L}(E,F)$. Let $\xi \in \mathcal{S}_1$ be a unit vector. Then, from **the definition of composition of functions**, Lemma 148 (*operator norm estimation*, for g and f), Lemma 116 (*equivalent definition of unit sphere*, $\|\xi\|_E = 1$), and **field properties of $\mathbb{R}$**, we have

$$\|(g \circ f)(\xi)\|_F = \|g(f(\xi))\|_F \leq |||g|||_{G,F}\, \|f(\xi)\|_F \leq |||g|||_{G,F}\, |||f|||_{F,E}\, \|\xi\|_E = |||g|||_{G,F}\, |||f|||_{F,E}.$$

Hence, from Lemma 144 (*finite operator norm is continuous*, $|||g|||_{G,F}\, |||f|||_{F,E}$ is a finite upper bound for $\|(g \circ f)(\mathcal{S}_1)\|_F$), $g \circ f$ belongs to $\mathcal{L}_c(E,F)$ and

$$|||g \circ f|||_{G,E} \leq |||g|||_{G,F}\, |||f|||_{F,E}.$$

□

Lemma 151 ***(complete normed vector space of continuous linear maps).*** *Let* $(E, \|\cdot\|_E)$ *and* $(F, \|\cdot\|_F)$ *be normed vector spaces. If* $(F, \|\cdot\|_F)$ *is complete, then the normed vector space* $\mathcal{L}_c(E,F)$ *is also complete (i.e. they are both Banach spaces).*

Proof. **Case $E = \{0_E\}$.** Then, from Lemma 91 (*linear map preserves zero*), $\mathcal{L}_c(E,F)$ is also the singleton $\{0_{\mathcal{L}(E,F)}\}$. From Definition 38 (*complete metric space*), and Lemma 33 (*stationary sequence is convergent*), singletons are trivially complete metric spaces since they possess only one sequence which is constant, hence stationary, hence convergent. Therefore, $\mathcal{L}_c(E,F)$ is complete.

Case $E \neq \{0_E\}$.

Pointwise limit. Let $d_{F,E}$ be the distance associated with norm $|||\cdot|||_{F,E}$. From Lemma 111 (*norm gives distance*), $(\mathcal{L}_c(E,F), d_{F,E})$ is a metric space. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $(\mathcal{L}_c(E,F), d_{F,E})$. Then, from Definition 34 (*Cauchy sequence*), Definition 109 (*distance associated with norm*), Theorem 147 (*normed vector space of continuous linear maps*, definition of $|||\cdot|||_{F,E}$), Definition 135 (*operator norm*), and Lemma 144 (*finite operator norm is continuous*, $|||f_p - f_q|||_{F,E}$ is lower than or equal to continuity constants), we have

$$\forall \varepsilon > 0,\, \exists N \in \mathbb{N},\, \forall p, q \in \mathbb{N}, \quad p, q \geq N \quad \Longrightarrow \quad \forall u \in E, \quad \|f_p(u) - f_q(u)\|_F \leq \varepsilon \|u\|_E. \tag{90}$$

Let $u \in E$. **Case $\boldsymbol{u \neq 0_E}$.** Let $\varepsilon' > 0$. From Definition 102 (*norm*, $\|\cdot\|_E$ is definite, contrapositive), $\|u\|_E \neq 0$ and from Equation (90) with $\varepsilon = \frac{\varepsilon'}{\|u\|_E}$, we have

$$\exists N \in \mathbb{N},\, \forall p, q \in \mathbb{N}, \quad p, q \geq N \quad \Longrightarrow \quad \|f_p(u) - f_q(u)\|_F \leq \varepsilon'.$$

Thus, from Definition 34 (*Cauchy sequence*, $(f_n(u))_{n\in\mathbb{N}}$ is a Cauchy sequence), and Definition 37 (*complete subset*, F is complete), let $f(u) = \lim_{n\to+\infty} f_n(u)$ be the limit in F.
Case $\boldsymbol{u = 0_E}$. Since from Lemma 91 (*linear map preserves zero*), we have for all $n \in \mathbb{N}$, $f_n(0_E) = 0_F$, let $f(0_E) = 0_F = \lim_{n\to+\infty} f_n(0_E)$.

Linearity. Let $u, v \in E$ and $\lambda, \mu \in \mathbb{K}$. From Definition 62 (*linear map*, for all $n \in \mathbb{N}$, f_n is a linear map), Lemma 128 (*vector addition is continuous*), Lemma 129 (*scalar multiplication is continuous*), and Lemma 44 (*compatibility of limit with continuous functions*), we have

$$\begin{aligned} f(\lambda u + \mu v) &= \lim_{n\to+\infty} f_n(\lambda u + \mu v) = \lim_{n\to+\infty} (\lambda f_n(u) + \mu f_n(v)) \\ &= \lim_{n\to+\infty} (\lambda f_n(u)) + \lim_{n\to+\infty} (\mu f_n(v)) = \lambda \lim_{n\to+\infty} f_n(u) + \mu \lim_{n\to+\infty} f_n(v) = \lambda f(u) + \mu f(v). \end{aligned}$$

Hence, from Lemma 92 (*linear map preserves linear combinations*), f belongs to $\mathcal{L}(E,F)$.

Continuity. In Equation (90), we consider a fixed $u \in E$ and we take the limit when q goes to $+\infty$. Thus, from Lemma 121 (*norm is continuous*), Lemma 44 (*compatibility of limit with continuous functions*), and Definition 88 (*inherited vector operations*, on $\mathcal{L}(E,F)$), we have

$$\forall \varepsilon > 0,\, \exists N \in \mathbb{N},\, \forall p \in \mathbb{N}, \quad p \geq N \quad \Longrightarrow \quad \forall u \in E, \quad \|(f_p - f)(u)\|_F \leq \varepsilon \|u\|_E. \tag{91}$$

Hence, from Definition 139 (*bounded linear map*, $f_p - f$ is bounded), Theorem 142 (*continuous linear map*, $f_p - f$ is continuous), Theorem 147 (*normed vector space of continuous linear maps*, $\mathcal{L}_c(E,F)$ is a space), and Definition 58 (*vector space*, $(\mathcal{L}_c(E,F), +)$ is an abelian group), we have $f = f_p - (f_p - f)$ belongs to $\mathcal{L}_c(E,F)$.

Limit for $\boldsymbol{|||\cdot|||_{F,E}}$. From Lemma 144 (*finite operator norm is continuous*, ε is a continuity constant for $f_p - f$), Equation (91) becomes

$$\forall \varepsilon > 0,\, \exists N \in \mathbb{N},\, \forall p \in \mathbb{N}, \quad p \geq N \quad \Longrightarrow \quad |||f_p - f|||_{F,E} \leq \varepsilon.$$

Hence, from Definition 109 (*distance associated with norm*, for norm $|||\cdot|||_{F,E}$), and Definition 26 (*convergent sequence*), the sequence $(f_n)_{n\in\mathbb{N}}$ is convergent in $\mathcal{L}_c(E,F)$ for the distance $d_{F,E}$.

Therefore, from Definition 37 (*complete subset*), the normed vector space $\mathcal{L}_c(E,F)$ is complete. □

Definition 152 *(topological dual).* Let $(E, \|\cdot\|_E)$ be a normed vector space. The set of continuous linear forms on E, denoted $E' = \mathcal{L}_c(E, \mathbb{K})$, is called the *topological dual of E*.

Definition 153 *(dual norm).* Let $(E, \|\cdot\|_E)$ be a normed vector space. The *dual norm associated with $\|\cdot\|_E$*, denoted $\|\cdot\|_{E'}$, is the operator norm $|||\cdot|||_{\mathbb{K},E}$ on $E' = \mathcal{L}_c(E, \mathbb{K})$ induced by norms $\|\cdot\|_E$ and $|\cdot|$ (absolute value over $\mathbb{K}$).

Lemma 154 ***(topological dual is complete normed vector space).*** *Let $(E, \|\cdot\|_E)$ be a normed vector space. Let E' be the topological dual of E. Let $\|\cdot\|_{E'}$ be the associated dual norm. Then, $(E', \|\cdot\|_{E'})$ is a complete normed vector space.*

Proof. Direct consequence of Definition 153 (*dual norm*), Theorem 147 (*normed vector space of continuous linear maps*), Lemma 151 (*complete normed vector space of continuous linear maps*), Lemma 105 (*K is normed vector space*), and the **completeness of $\mathbb{K}$**. □

Definition 155 ***(bra-ket notation).*** Let $(E, \|\cdot\|_E)$ be a normed vector space. A continuous linear form $\varphi \in E'$ is a *bra*, denoted $\langle\varphi|$. A vector $u \in E$ is a *ket*, denoted $|u\rangle$. In *bra-ket notation* (or *Dirac notation*, or *duality pairing*), the application $\varphi(u)$ is denoted $\langle\varphi|u\rangle_{E',E}$.

Lemma 156 ***(bra-ket is bilinear map).*** *Let $(E, \|\cdot\|_E)$ be a normed vector space. Then, $\langle\cdot|\cdot\rangle_{E',E}$ is a bilinear map from $E' \times E$ to $\mathbb{K}$.*

Proof. From Lemma 87 (*product is space*), and Lemma 154 (*topological dual is complete normed vector space*), $E' \times E$ is a space. From Lemma 101 (*K is space*), $\mathbb{K}$ is a space. From Definition 155 (*bra-ket notation*), $\langle\cdot|\cdot\rangle_{E',E}$ is a mapping from $E' \times E$ to $\mathbb{K}$.

Let $\lambda, \mu \in \mathbb{R}$ be scalars. Let $\varphi, \psi \in E'$ be continuous linear forms on E (i.e. bras). Let $u, v \in E$ be vectors (i.e. kets). Then, from Definition 155 (*bra-ket notation*), and Definition 88 (*inherited vector operations*, on E'), we have

$$\langle\lambda\varphi + \mu\psi|u\rangle_{E',E} = (\lambda\varphi + \mu\psi)(u) = \lambda\varphi(u) + \mu\psi(u) = \lambda\,\langle\varphi|u\rangle_{E',E} + \mu\,\langle\psi|u\rangle_{E',E}\,.$$

Moreover, from Definition 155 (*bra-ket notation*), and Definition 62 (*linear map*, φ is linear), we have

$$\langle\varphi|\lambda u + \mu v\rangle_{E',E} = \varphi(\lambda u + \mu v) = \lambda\varphi(u) + \mu\varphi(v) = \lambda\,\langle\varphi|u\rangle_{E',E} + \mu\,\langle\varphi|v\rangle_{E',E}\,.$$

Therefore, from Definition 65 (*bilinear map*), $\langle\cdot|\cdot\rangle_{E',E}$ is left and right linear, hence bilinear. □

4.4.1.2 Bounded bilinear form

Definition 157 ***(bounded bilinear form).*** Let $(E, \|\cdot\|_E)$ be a normed vector space. A bilinear form $\varphi \in \mathcal{L}_2(E)$ is *bounded* iff

$$\exists C \geq 0,\ \forall u, v \in E, \quad |\varphi(u, v)| \leq C\,\|u\|_E\,\|v\|_E\,. \tag{92}$$

Then, C is called *continuity constant of* φ.

Lemma 158 ***(representation for bounded bilinear form).*** *Let $(E, \|\cdot\|_E)$ be a normed vector space. Let $\varphi \in \mathcal{L}_2(E)$ be a bilinear form on E. Assume that φ is bounded. Then, there exists a unique continuous linear map $A \in \mathcal{L}_c(E, E')$ such that*

$$\forall u, v \in E, \quad \varphi(u, v) = \langle A(u)|v\rangle_{E',E} = A(u)(v). \tag{93}$$

Moreover, for all C continuity constant of φ, we have

$$\|A\|_{E',E} \leq C. \tag{94}$$

Proof. From Definition 67 (*set of bilinear forms*), and Definition 66 (*bilinear form*), φ is a bilinear map.

Existence. Let $u \in E$ be a vector. Let $A_u : E \to \mathbb{R}$ be the mapping defined by

$$\forall v \in E, \quad A_u(v) = \varphi(u, v).$$

Let $\lambda, \lambda' \in \mathbb{R}$ be scalars. Let $v, v' \in E$ be vectors. Then, from Definition 65 (*bilinear map*, $\underline{\varphi \text{ is right linear}}$), we have

$$A_u(\lambda v + \lambda' v') = \varphi(u, \lambda v + \lambda' v') = \lambda \varphi(u, v) + \lambda' \varphi(u, v') = \lambda A_u(v) + \lambda' A_u(v').$$

Hence, from Lemma 92 (*linear map preserves linear combinations*), and Definition 64 (*linear form*), A_u is a linear form on E.

Let $v \in E$ be a vector. From Definition 157 (*bounded bilinear form*, $\underline{\text{for } \varphi}$), let $C \geq 0$ such that, for all $u', v' \in E$, we have $|\varphi(u', v')| \leq C \left\| u' \right\|_E \left\| v' \right\|_E$. Let $C_u = C \left\| u \right\|_E$. Then, from Lemma 107 (*norm is nonnegative*), and **ordered field properties of $\mathbb{R}$**, we have $C_u \geq 0$ and

$$|A_u(v)| = |\varphi(u, v)| \leq C \left\| u \right\|_E \left\| v \right\|_E = C_u \left\| v \right\|_E .$$

Hence, from Definition 139 (*bounded linear map*, $\underline{A_u \text{ is bounded}}$), Definition 152 (*topological dual*), Definition 153 (*dual norm*), and Lemma 144 (*finite operator norm is continuous*, $\underline{A_u \in E'}$), we have

$$\left\| A_u \right\|_{E'} \leq C_u = C \left\| u \right\|_E .$$

Let $A : E \to E'$ be the mapping defined by, for all $u \in E$, $A(u) = A_u$, i.e.

$$\forall u, v \in E, \quad \langle A(u)|v \rangle_{E',E} = A(u)(v) = A_u(v) = \varphi(u, v).$$

Let $\lambda, \lambda' \in \mathbb{R}$ be scalars. Let $u, u', v \in E$ be vectors. Then, from Definition 65 (*bilinear map*, $\underline{\varphi \text{ is left linear}}$), Definition 88 (*inherited vector operations*, $\underline{\text{on } E'}$), and Lemma 154 (*topological dual is complete normed vector space*, $\underline{E' \text{ is space}}$), we have

$$\begin{array}{rcl}
A(\lambda u + \lambda' u')(v) & = & A_{(\lambda u + \lambda' u')}(v) \\
& = & \varphi(\lambda u + \lambda' u', v) \\
& = & \lambda \varphi(u, v) + \lambda' \varphi(u', v) \\
& = & \lambda A_u(v) + \lambda' A'_u(v) \\
& = & \lambda A(u)(v) + \lambda' A(u')(v) \\
& = & (\lambda A(u) + \lambda' A(u'))(v).
\end{array}$$

Hence, from Lemma 92 (*linear map preserves linear combinations*), A is a linear map from E to E'.

Let $\mathcal{S}_1$ be the unit sphere of E. Let $\xi \in \mathcal{S}_1$ be a unit vector. Then, from Lemma 116 (*equivalent definition of unit sphere*, $\underline{\left\| \xi \right\|_E = 1}$), we have

$$\left\| A(\xi) \right\|_{E'} = \left\| A_\xi \right\|_{E'} \leq C \left\| \xi \right\|_E = C.$$

Hence, from Lemma 144 (*finite operator norm is continuous*, $\underline{C \text{ is a finite upper bound for } \left\| A(\mathcal{S}_1) \right\|_{E'}}$), A belongs to $\mathcal{L}_c(E, E')$ and $|||A|||_{E',E} \leq C$.

Uniqueness. Let $A, A' \in \mathcal{L}_c(E, E')$ be continuous linear maps such that

$$\forall u, v \in E, \quad \varphi(u, v) = \langle A(u)|v \rangle_{E',E} = \langle A'(u)|v \rangle_{E',E} .$$

From Theorem 147 (*normed vector space of continuous linear maps*), Definition 104 (*normed vector space*, $\underline{\mathcal{L}_c(E, E') \text{ is a space}}$), and Definition 70 (*vector subtraction*), let $B = A - A' \in \mathcal{L}_c(E, E')$. Let $u, v \in E$ be vectors. Then, from Lemma 156 (*bra-ket is bilinear map*), Lemma 101 (*K is space*), and Definition 58 (*vector space*, $\underline{(\mathbb{K}, +) \text{ is an abelian group}}$), we have

$$\langle B(u)|v \rangle_{E',E} = \langle A(u)|v \rangle_{E',E} - \langle A'(u)|v \rangle_{E',E} = \varphi(u, v) - \varphi(u, v) = 0.$$

Thus, from Definition 155 (*bra-ket notation*), $B(u) = 0_{E'}$, and then $B = 0_{\mathcal{L}_c(E,E')}$. Hence, from Definition 58 (*vector space*, $\underline{(\mathcal{L}_c(E, E'), +) \text{ is an abelian group}}$), $A = A'$. $\square$

Definition 159 ***(coercive bilinear form).*** Let $(E, \|\cdot\|_E)$ be a real normed vector space. A bilinear form $\varphi \in \mathcal{L}_2(E)$ is *coercive* (or *elliptic*) iff

$$\exists \alpha > 0, \forall u \in E, \quad \varphi(u, u) \geq \alpha \|u\|_E^2 . \tag{95}$$

Then, α is called *coercivity constant of* φ.

Lemma 160 ***(coercivity constant is less than continuity constant).*** *Let $(E, \|\cdot\|_E)$ be a real normed vector space. Let $\varphi \in \mathcal{L}_2(E)$ be a bilinear form on E. Assume that φ is continuous with constant $C \geq 0$, and coercive with constant $\alpha > 0$. Then, $\alpha \leq C$.*

Proof. Let $u \in E$ be a vector. Assume that $u \neq 0_E$. Then, from Definition 102 (*norm*, $\|\cdot\|_E$ is definite, contrapositive), and Lemma 107 (*norm is nonnegative*, for $\|\cdot\|_E$), we have $\|u\|_E > 0$. From Definition 159 (*coercive bilinear form*), **properties of the absolute value on** $\mathbb{R}$, and Definition 157 (*bounded bilinear form*, with $v = u$), we have

$$\alpha \|u\|_E^2 \leq \varphi(u, u) \leq |\varphi(u, u)| \leq C \|u\|_E^2 .$$

Hence, from **ordered field properties of** $\mathbb{R}$, $\alpha \leq C$. □

4.5 Inner product space

Definition 161 ***(inner product).*** Let G be a real space. A mapping $(\cdot, \cdot)_G : G \times G \to \mathbb{R}$ is an *inner product on* G iff it is a bilinear form on G that is symmetric, nonnegative, and definite:

$$\forall u, v \in G, \qquad (u, v)_G = (v, u)_G ; \tag{96}$$
$$\forall u \in G, \qquad (u, u)_G \geq 0; \tag{97}$$
$$\forall u \in G, \qquad (u, u)_G = 0 \quad \Longrightarrow \quad u = 0_G. \tag{98}$$

Remark 162. Note that the symmetry property (96) implies the equivalence between left additivity (46) and right additivity (47) in the definition of a bilinear map.

Remark 163. Most results below are valid on a semi-inner space in which the definite property (98) is dropped. The associated norm is then a semi-norm (the separation property is dropped).

Remark 164. In the case of a complex space, the symmetry property becomes a conjugate symmetry property. In the sequel, we specify that the space is real only in the case where the very same statement does not hold in a complex space. When proofs differ, they are only given in the real case.

Definition 165 ***(inner product space).*** $(G, (\cdot, \cdot)_G)$ is an *inner product space* (or *pre-Hilbert space*) iff G is a space and $(\cdot, \cdot)_G$ is an inner product on G.

Lemma 166 ***(inner product subspace).*** *Let $(G, (\cdot, \cdot)_G)$ be an inner product space. Let F be a subspace of G. Then, F equipped with the restriction to F of the inner product $(\cdot, \cdot)_G$ is an inner product space.*

Proof. Direct consequence of Definition 74 (*subspace*, F is a subset of G and F is a space), Definition 161 (*inner product*, the restriction of $(\cdot, \cdot)_G$ to F is trivially an inner product on F), and Definition 165 (*inner product space*). □

Lemma 167 ***(inner product with zero is zero).*** *Let $(G, (\cdot, \cdot)_G)$ be an inner product space. Then,*

$$\forall u \in G, \quad (0, u)_G = (u, 0)_G = 0. \tag{99}$$

Proof. Let $u \in G$. From Definition 161 (*inner product*, $(\cdot,\cdot)_G$ is symmetric and a bilinear map), Definition 58 (*vector space*, $(G,+)$ is an abelian group), Definition 70 (*vector subtraction*), Definition 65 (*bilinear map*, $(\cdot,\cdot)_G$ is right linear), and **field properties of** $\mathbb{R}$, we have

$$(0_G, u)_G = (u, 0_G)_G = (u, 0_G - 0_G)_G = (u, 0_G)_G - (u, 0_G)_G = 0.$$

□

Lemma 168 ***(square expansion plus).*** *Let* $(G, (\cdot,\cdot)_G)$ *be a real inner product space. Then,*

$$\forall u, v \in G, \quad (u+v, u+v)_G = (u,u)_G + 2\,(u,v)_G + (v,v)_G\,. \tag{100}$$

Proof. Let $u, v \in G$ be vectors. From Definition 161 (*inner product*, $(\cdot,\cdot)_G$ is a bilinear map and symmetric), Definition 65 (*bilinear map*), and **field properties of** $\mathbb{R}$, we have

$$(u+v, u+v)_G = (u,u)_G + (u,v)_G + (v,u)_G + (v,v)_G = (u,u)_G + 2\,(u,v)_G + (v,v)_G\,.$$

□

Lemma 169 ***(square expansion minus).*** *Let* $(G, (\cdot,\cdot)_G)$ *be a real inner product space. Then,*

$$\forall u, v \in G, \quad (u-v, u-v)_G = (u,u)_G - 2\,(u,v)_G + (v,v)_G\,. \tag{101}$$

Proof. Let $u, v \in G$ be vectors. From Definition 70 (*vector subtraction*), Lemma 168 (*square expansion plus*), Definition 161 (*inner product*, $(\cdot,\cdot)_G$ is a bilinear map), Definition 65 (*bilinear map*), and **field properties of** $\mathbb{R}$, we have

$$\begin{aligned}
(u-v, u-v)_G &= (u+(-v), u+(-v))_G \\
&= (u,u)_G + 2\,(u,-v)_G + (-v,-v)_G \\
&= (u,u)_G - 2\,(u,v)_G + (v,v)_G\,.
\end{aligned}$$

□

Lemma 170 ***(parallelogram identity).*** *Let* $(G, (\cdot,\cdot)_G)$ *be an inner product space. Then,*

$$\forall u, v \in G, \quad (u+v, u+v)_G + (u-v, u-v)_G = 2\left((u,u)_G + (v,v)_G\right). \tag{102}$$

Proof. Let $u, v \in G$ be vectors. From Lemma 168 (*square expansion plus*), Lemma 169 (*square expansion minus*), and **field properties of** $\mathbb{R}$, we have

$$\begin{aligned}
(u+v, u+v)_G + (u-v, u-v)_G &= (u,u)_G + 2\,(u,v)_G + (v,v)_G + (u,u)_G - 2\,(u,v)_G + (v,v)_G \\
&= 2\left((u,u)_G + (v,v)_G\right).
\end{aligned}$$

□

Lemma 171 ***(Cauchy–Schwarz inequality).*** *Let* $(G, (\cdot,\cdot)_G)$ *be a real inner product space. Then,*

$$\forall u, v \in G, \quad \left((u,v)_G\right)^2 \leq (u,u)_G\,(v,v)_G\,. \tag{103}$$

Proof. Let $u, v \in G$ be vectors. Let $\lambda \in \mathbb{R}$ be a scalar. From Lemma 168 (*square expansion plus*), Definition 161 (*inner product*, $(\cdot,\cdot)_G$ is a bilinear map), Definition 65 (*bilinear map*), and **field properties of** $\mathbb{R}$, we have

$$(u+\lambda v, u+\lambda v)_G = \lambda^2\,(v,v)_G + 2\lambda\,(u,v)_G + (u,u)_G\,.$$

Let $P(X) = (v,v)_G\,X^2 + 2\,(u,v)_G\,X + (u,u)_G$. It is a quadratic polynomial with real coefficients. From Definition 161 (*inner product*, $(\cdot,\cdot)_G$ is nonnegative), the associated polynomial function P

is nonnegative. Hence, since **a quadratic polynomial function has a constant sign iff its discriminant is nonpositive**, we have

$$4((u,v)_G)^2 - 4\,(v,v)_G\,(u,u)_G \leq 0.$$

Therefore, from **ordered field properties of $\mathbb{R}$**, we have

$$((u,v)_G)^2 \leq (u,u)_G\,(v,v)_G\,.$$

□

Definition 172 ***(square root of inner square).*** Let $(G,(\cdot,\cdot)_G)$ be an inner product space. The associated *square root of inner square* is the mapping $N_G : G \to \mathbb{R}$ defined by

$$\forall u \in G, \quad N_G(u) = \sqrt{(u,u)_G}. \tag{104}$$

Remark 173. Mapping N_G is well defined thanks to the nonnegativeness of the inner product. It will be proved below to be a norm.

Lemma 174 ***(squared norm).*** *Let $(G,(\cdot,\cdot)_G)$ be an inner product space. Then,*

$$\forall u \in G, \quad N_G(u)^2 = (u,u)_G\,. \tag{105}$$

Proof. Direct consequence of Definition 172 (*square root of inner square*), Definition 161 (*inner product*, $(\cdot,\cdot)_G$ is nonnegative), and **properties of square and square root functions in $\mathbb{R}^+$**. □

Lemma 175 ***(Cauchy–Schwarz inequality with norms).*** *Let $(G,(\cdot,\cdot)_G)$ be an inner product space. Then,*

$$\forall u,v \in G, \quad |\,(u,v)_G\,| \leq N_G(u)\,N_G(v). \tag{106}$$

Proof. Direct consequence of Lemma 171 (*Cauchy–Schwarz inequality*), Definition 172 (*square root of inner square*), Definition 161 (*inner product*, $(\cdot,\cdot)_G$ is nonnegative), and **compatibility of the square root function with comparison in $\mathbb{R}^+$**. □

Lemma 176 ***(triangle inequality).*** *Let $(G,(\cdot,\cdot)_G)$ be an inner product space. Then,*

$$\forall u,v \in G, \quad N_G(u+v) \leq N_G(u) + N_G(v). \tag{107}$$

Proof. Let $u,v \in G$ be vectors. From Lemma 174 (*squared norm*), Lemma 168 (*square expansion plus*), Lemma 175 (*Cauchy–Schwarz inequality with norms*), and **field properties of $\mathbb{R}$**, we have

$$\begin{aligned}
(N_G(u+v))^2 &= (u+v,u+v)_G \\
&= (u,u)_G + 2\,(u,v)_G + (v,v)_G \\
&\leq (N_G(u))^2 + 2N_G(u)\,N_G(v) + (N_G(v))^2 \\
&= (N_G(u) + N_G(v))^2\,.
\end{aligned}$$

Therefore, from Definition 161 (*inner product*, $(\cdot,\cdot)_G$ is nonnegative), and **compatibility of the square function with comparison in $\mathbb{R}^+$**, we have

$$N_G(u+v) \leq N_G(u) + N_G(v).$$

□

Lemma 177 ***(inner product gives norm).*** *Let $(G,(\cdot,\cdot)_G)$ be an inner product space. Let $\|\cdot\|_G$ be the associated square root of inner square. Then, $(G,\|\cdot\|_G)$ is a normed vector space.*

Proof. From Definition 172 (*square root of inner square*), and **nonnegativeness of the square root function in $\mathbb{R}^+$**, $\|\cdot\|_G$ is nonnegative.

From Definition 172 (*square root of inner square*), **definiteness of the square root function in $\mathbb{R}^+$**, and Definition 161 (*inner product*, $(\cdot,\cdot)_G$ is definite), $\|\cdot\|_G$ is definite.

Let $\lambda \in \mathbb{K}$ be a scalar. Let $u \in G$ be a vector. From Definition 172 (*square root of inner square*), Definition 161 (*inner product*, $(\cdot,\cdot)_G$ is a bilinear map), **multiplicativity of the square root function in $\mathbb{R}^+$**, and since **for all $x \in \mathbb{R}$, $\sqrt{x^2} = |x|$**, we have

$$\|\lambda u\|_G = \sqrt{(\lambda u, \lambda u)_G} = \sqrt{\lambda^2 (u,u)_G} = \sqrt{\lambda^2}\sqrt{(u,u)_G} = |\lambda|\ \|u\|_G\,.$$

Thus, $\|\cdot\|_G$ is absolutely homogeneous of degree 1.

From Lemma 176 (*triangle inequality*), $\|\cdot\|_G$ satisfies triangle inequality.

Therefore, from Definition 102 (*norm*), $\|\cdot\|_G$ is a norm over G, and from Definition 104 (*normed vector space*), $(G, \|\cdot\|_G)$ is a normed vector space. □

Remark 178. Norm $\|\cdot\|_G$ is called *norm associated with inner product* $(\cdot,\cdot)_G$.

4.5.1 Orthogonal projection

Definition 179 *(convex subset).* Let E be a real space. Let $K \subset E$. K is *convex* iff

$$\forall u, v \in K, \forall \theta \in [0,1], \quad \theta u + (1-\theta) v \in K. \tag{108}$$

Theorem 180 *(orthogonal projection onto nonempty complete convex).* *Let $(G, (\cdot,\cdot)_G)$ be a real inner product space. Let $\|\cdot\|_G$ be the norm associated with inner product $(\cdot,\cdot)_G$. Let d_G be the distance associated with norm $\|\cdot\|_G$. Let $K \subset G$ be a nonempty convex subset which is complete for distance d_G. Then, for all $u \in G$, there exists a unique $v \in K$ such that*

$$\|u - v\|_G = \min_{w \in K} \|u - w\|_G\,. \tag{109}$$

Proof. Let $u \in G$. From Lemma 107 (*norm is nonnegative*, for $\|\cdot\|_G$), function $w \mapsto \|u-w\|_G$ from K to $\mathbb{R}$ admits 0 as finite lower bound. Thus, from Lemma 13 (*finite infimum discrete*), $\delta = \inf_{w\in K}\{\|u-w\|_G\}$ is finite and there exists a sequence $(w_n)_{n\in\mathbb{N}}$ in K such that for all $n \in \mathbb{N}$, $\|u - w_n\|_G < \delta + \frac{1}{n+1}$.

From Definition 165 (*inner product space*), G is a space.

Existence. Let $p, q \in \mathbb{N}$. Let $a = u - w_q$ and $b = u - w_p$. From Definition 102 (*norm*, $\|\cdot\|_G$ is absolutely homogeneous of degree 1), Definition 71 (*scalar division*), Definition 58 (*vector space*, $(G,+)$ is an abelian group), Lemma 174 (*squared norm*), and Lemma 170 (*parallelogram identity*, for $\|\cdot\|_G$), we have

$$\begin{aligned} 4\left\|u - \frac{w_q + w_p}{2}\right\|_G^2 + \|w_p - w_q\|_G^2 &= \|a+b\|_G^2 + \|a-b\|_G^2 \\ &= 2\|a\|_G^2 + 2\|b\|_G^2 \\ &= 2\|u-w_q\|_G^2 + 2\|u-w_p\|_G^2\,. \end{aligned}$$

From Definition 179 (*convex subset*), $\frac{w_q+w_p}{2}$ belongs to K. Thus, from Definition 9 (*infimum*, δ is a lower bound for $\{\|u-w\|_G \mid w \in K\}$), and **field properties of $\mathbb{R}$**, we have

$$\begin{aligned} \|w_p - w_q\|_G^2 &= -4\left\|u - \frac{w_q+w_p}{2}\right\|_G^2 + 2\|u-w_q\|_G^2 + 2\|u-w_p\|_G^2 \\ &< -4\delta^2 + 2\left(\delta + \frac{1}{q+1}\right)^2 + 2\left(\delta + \frac{1}{p+1}\right)^2 \\ &= \frac{4\delta}{q+1} + \frac{2}{(q+1)^2} + \frac{4\delta}{p+1} + \frac{2}{(p+1)^2}. \end{aligned}$$

Let $\varepsilon > 0$. Let $\eta = \max\left(\frac{16\delta}{\varepsilon^2}, \frac{2\sqrt{2}}{\varepsilon}\right)$. From **the definition of the max function**, and **ordered field properties of $\mathbb{R}$**, $\eta > 0$. Let $N = \lceil \eta \rceil - 1$. From **the definition of the ceiling function**, $N \geq 0$ and $N \geq \eta - 1$. Assume that $p, q \geq N$. Then, from **ordered field properties of $\mathbb{R}$**, we have $p, q \geq \eta - 1$ and $\frac{4\delta}{q+1}, \frac{2}{(q+1)^2}, \frac{4\delta}{p+1}, \frac{2}{(p+1)^2} \leq \frac{\varepsilon^2}{4}$. Thus, from **field properties of $\mathbb{R}$**, Lemma 107 (*norm is nonnegative*, for $\|\cdot\|_G$), and **compatibility of the square root function with comparison in $\mathbb{R}^+$**, we have $\|w_p - w_q\|_G \leq \varepsilon$. Hence, from Definition 34 (*Cauchy sequence*), $(w_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in K.

From hypothesis, and Definition 37 (*complete subset*, K is complete), the sequence $(w_n)_{n\in\mathbb{N}}$ is convergent in K. Let $v \in K$ be its limit. From Lemma 121 (*norm is continuous*, for $\|\cdot\|_G$), Lemma 44 (*compatibility of limit with continuous functions*), and Definition 14 (*minimum*), we have

$$\|u - v\|_G = \lim_{n\to+\infty} \|u - w_n\|_G = \delta = \min_{w\in K} \|u - w\|_G .$$

Uniqueness. Let $v, v' \in K$ such that $\|u - v\|_G = \|u - v'\|_G = \delta$. Let $a = u - v'$, $b = u - v$, and $v'' = \frac{v'+v}{2}$. Then, from Definition 102 (*norm*, $\|\cdot\|_G$ is absolutely homogeneous of degree 1), Definition 71 (*scalar division*), Definition 58 (*vector space*, $(G, +)$ is an abelian group), and Lemma 170 (*parallelogram identity*, for $\|\cdot\|_G$), we have

$$\begin{aligned} 4\|u - v''\|_G^2 + \|v - v'\|_G^2 &= \|a + b\|_G^2 + \|a - b\|_G^2 \\ &= 2\|a\|_G^2 + 2\|b\|_G^2 \\ &= 2\|u - v'\|_G^2 + 2\|u - v\|_G^2 \\ &= 4\delta^2. \end{aligned}$$

From Definition 179 (*convex subset*), v'' belongs to K. Thus, from Definition 9 (*infimum*, δ is a lower bound for $\{\|u - w\|_G \mid w \in K\}$), and **field properties of $\mathbb{R}$**, we have

$$0 \leq \|v - v'\|_G^2 = -4\|u - v''\|_G^2 + 4\delta^2 \leq -4\delta^2 + 4\delta^2 = 0.$$

Hence, from Lemma 107 (*norm is nonnegative*, for $\|\cdot\|_G$), and **compatibility of square root function with comparison in $\mathbb{R}^+$**, $\|v - v'\|_G = 0$. Therefore, from Definition 102 (*norm*, $\|\cdot\|_G$ is definite), Definition 70 (*vector subtraction*), and Definition 58 (*vector space*, $(G, +)$ is an abelian group), we have $v - v' = 0_G$ and $v = v'$. □

Lemma 181 *(characterization of orthogonal projection onto convex).* *Let $(G, (\cdot,\cdot)_G)$ be a real inner product space. Let $\|\cdot\|_G$ be the norm associated with inner product $(\cdot,\cdot)_G$. Let $K \subset G$ be a nonempty convex subset. Then, for all $u \in G$, for all $v \in K$,*

$$\|u - v\|_G = \inf_{w\in K} \|u - w\|_G \iff \forall w \in K, \quad (u - v, w - v)_G \leq 0. \tag{110}$$

Proof. Let $u \in G$ and $v \in K$ be vectors.

"Left" implies "right". Assume that $\|u - v\|_G = \inf_{w\in K} \|u - w\|_G$. Let $w \in K$. Let $\theta \in (0, 1]$. From Definition 179 (*convex subset*), $\theta w + (1-\theta)v$ belongs to K. Thus, from Definition 9 (*infimum*, $\|u - v\|_G$ is a lower bound for $\{\|u - w\|_G \mid w \in K\}$), **compatibility of the square function with comparison in $\mathbb{R}^+$**, Definition 165 (*inner product space*, G is a space), Definition 58 (*vector space*, $(G, +)$ is an abelian group and scalar multiplication is compatible with scalar addition), 70 (*vector subtraction*), Lemma 174 (*squared norm*, for $(\cdot,\cdot)_G$), Lemma 168 (*square expansion plus*, for $(\cdot,\cdot)_G$), Definition 161 (*inner product*, $(\cdot,\cdot)_G$ is a bilinear map), Definition 65 (*bilinear map*), and Lemma 168 (*square expansion plus*), we have

$$\begin{aligned} \|u - v\|_G^2 &\leq \|u - (\theta w + (1-\theta)v)\|_G^2 \\ &= \|(u - v) + \theta(v - w)\|_G^2 \\ &= \|u - v\|_G^2 - 2\theta\,(u - v, w - v)_G + \theta^2\,\|v - w\|_G^2 . \end{aligned}$$

Let $a = (u - v, w - v)_G$ and $b = \|v - w\|_G^2$. Then, from **ordered field properties of $\mathbb{R}$ (with $\theta > 0$)**, we have

$$\forall \theta \in (0, 1], \quad 2a \leq \theta b.$$

Assume that $b = 0$. Then, from **ordered field properties of $\mathbb{R}$**, we have $(u - v, w - v)_G = a \leq 0$. Conversely, assume now that $b \neq 0$. Then, from **nonnegativeness of the square function**, $b > 0$. Assume that $a > 0$. Let $\theta = \min(1, \frac{a}{b})$. From **the definition of the min function**, and **ordered field properties of $\mathbb{R}$**, we have $\theta \leq \frac{a}{b}$ and $0 < \theta \leq 1$. Thus, $2a \leq \theta b \leq a$. Hence, from **ordered field properties of $\mathbb{R}$**, $a \leq 0$. Which is impossible. Therefore, we have $(u - v, w - v)_G = a \leq 0$.

"Right" implies "left". Conversely, assume now that, for all $w \in K$, $(u - v, w - v)_G \leq 0$. Let $w \in K$. Then, from Definition 165 (*inner product space*, G is a space), Definition 58 (*vector space*, $(G, +)$ is an abelian group), Definition 70 (*vector subtraction*), Lemma 174 (*squared norm*, for $(\cdot, \cdot)_G$), Lemma 168 (*square expansion plus*, for $(\cdot, \cdot)_G$), Definition 161 (*inner product*, $(\cdot, \cdot)_G$ is a bilinear map), Definition 65 (*bilinear map*, $(\cdot, \cdot)_G$ is right linear), and **nonnegativeness of the square function in $\mathbb{R}^+$**, we have

$$\begin{aligned}
\|u - w\|_G^2 &= \|(u - v) + (v - w)\|_G^2 \\
&= \|u - v\|_G^2 + 2\,(u - v, v - w)_G + \|v - w\|_G^2 \\
&\geq \|u - v\|_G^2 - 2\,(u - v, w - v)_G \\
&\geq \|u - v\|_G^2 .
\end{aligned}$$

Hence, from Lemma 107 (*norm is nonnegative*, for $\|\cdot\|_G$), and **compatibility of the square root function with comparison in $\mathbb{R}^+$**, we have $\|u - v\|_G \leq \|u - w\|_G$. Therefore, from Lemma 15 (*finite minimum*), and Definition 14 (*minimum*), we have

$$\|u - v\|_G = \min_{w \in K} \|u - w\|_G = \inf_{w \in K} \|u - w\|_G .$$

□

Lemma 182 *(subspace is convex).* *Let E be a real space. Let F be a subspace of E. Then, F is a convex subset of E.*

Proof. Let $u, v \in F$ be vectors in the subspace. Let $\theta \in [0, 1]$. Then, from Lemma 79 (*closed under linear combination is subspace*), the linear combination $w = \theta u + (1 - \theta) v$ belongs to F. Therefore, from Definition 179 (*convex subset*), F is a convex subset of E. □

Theorem 183 *(orthogonal projection onto complete subspace).* *Let $(G, (\cdot, \cdot)_G)$ be a real inner product space. Let $\|\cdot\|_G$ be the norm associated with inner product $(\cdot, \cdot)_G$. Let d_G be the distance associated with norm $\|\cdot\|_G$. Let F be a subspace of G which is complete for distance d_G. Then, for all $u \in G$, there exists a unique $v \in F$ such that*

$$\|u - v\|_G = \min_{w \in F} \|u - w\|_G . \tag{111}$$

Proof. Direct consequence of Definition 74 (*subspace*, F is vector space), Definition 58 (*vector space*, $F \ni 0_G$ is nonempty), Lemma 182 (*subspace is convex*), and Theorem 180 (*orthogonal projection onto nonempty complete convex*, F is a nonempty convex subset of G which is complete for distance d_G). □

Definition 184 *(orthogonal projection onto complete subspace).* Assume hypotheses of Theorem 183 (*orthogonal projection onto complete subspace*). The mapping $P_F : G \to F$ associating to any vector of G the unique vector of F satisfying (111) is called *orthogonal projection onto F*.

Lemma 185 ***(characterization of orthogonal projection onto subspace).*** *Let $(G, (\cdot,\cdot)_G)$ be a real inner product space. Let $\|\cdot\|_G$ be the norm associated with inner product $(\cdot,\cdot)_G$. Let F be a subspace of G. Then, for all $u \in G$, for all $v \in F$,*

$$\|u - v\|_G = \inf_{w\in F} \|u - w\|_G \quad \Longleftrightarrow \quad \forall w \in F, \quad (v, w)_G = (u, w)_G\,. \tag{112}$$

Proof. Let $u \in G$ and $v \in F$ be vectors.

"Left" implies "right". Assume that $\|u - v\|_G = \inf_{w\in F} \|u - w\|_G$. Then, from Definition 74 (*subspace*, F is vector space), Definition 58 (*vector space*, $F \ni 0_G$ is nonempty), Lemma 182 (*subspace is convex*), and Lemma 181 (*characterization of orthogonal projection onto convex*, F is a nonempty convex subset), we have for all $w \in F$, $(u - v, w - v)_G \leq 0$. Let $w \in F$. Let $w' = w + v$. Then, from Definition 74 (*subspace*, F is a space), Definition 58 (*vector space*, $(F, +)$ is an abelian group), and Definition 70 (*vector subtraction*), w' belongs to F and $w = w' - v$. Thus, we have $(u - v, w)_G = (u - v, w' - v)_G \leq 0$. Similarly, $w'' = -w + v$ belongs to F and $(u - v, -w)_G = (u - v, w'' - v)_G \leq 0$. Hence, from Definition 161 (*inner product*, $(\cdot,\cdot)_G$ is a bilinear map) Definition 65 (*bilinear map*, $(\cdot,\cdot)_G$ is right linear), and **ordered field properties of $\mathbb{R}$**, we have $(u - v, w)_G = 0$. Therefore, from Definition 70 (*vector subtraction*), and Definition 65 (*bilinear map*, $(\cdot,\cdot)_G$ is left linear), we have $(v, w)_G = (u, w)_G$.

"Right" implies "left". Conversely, assume now that for all $w \in F$, $(v, w)_G = (u, w)_G$. Let $w \in F$. Let $w' = w - v$. Then, from Definition 74 (*subspace*, F is a space), and Definition 58 (*vector space*, $(F, +)$ is an abelian group), w' belongs to F. Hence, from Definition 161 (*inner product*, $(\cdot,\cdot)_G$ is a bilinear map), Definition 65 (*bilinear map*, $(\cdot,\cdot)_G$ is left linear), hypothesis, and **ordered field properties of $\mathbb{R}$**, we have

$$(u - v, w - v)_G = (u - v, w')_G = (u, w')_G - (v, w')_G = 0 \leq 0.$$

Therefore, from Definition 74 (*subspace*, F is vector space), Definition 58 (*vector space*, $F \ni 0_G$ is nonempty), Lemma 182 (*subspace is convex*), and Lemma 181 (*characterization of orthogonal projection onto convex*, F is a nonempty convex subset), we have $\|u - v\|_G = \inf_{w\in F} \|u - w\|_G$. □

Lemma 186 ***(orthogonal projection is continuous linear map).*** *Assume hypotheses of Theorem 183 (*orthogonal projection onto complete subspace*). Then, the orthogonal projection P_F is a 1-Lipschitz continuous linear map from G to F.*

Proof. From Definition 184 (*orthogonal projection onto complete subspace*), and Theorem 183 (*orthogonal projection onto complete subspace*), P_F effectively defines a mapping from G to F.

Linearity. Let $u', u'' \in G$. Let $\lambda', \lambda'' \in \mathbb{R}$. From Definition 74 (*subspace*, F is a vector space), and Definition 58 (*vector space*, G and F are closed under vector operations), $\lambda' u' + \lambda'' u''$ belongs to G and $\lambda' P_F(u') + \lambda'' P_F(u'')$ belongs to F. Let $w \in F$. From Definition 161 (*inner product*, $(\cdot,\cdot)_G$ is a bilinear map), Definition 65 (*bilinear map*, $(\cdot,\cdot)_G$ is left linear), and Lemma 185 (*characterization of orthogonal projection onto subspace*), we have

$$\begin{aligned}
(\lambda' P_F(u') + \lambda'' P_F(u''), w)_G &= \lambda' (P_F(u'), w)_G + \lambda'' (P_F(u''), w)_G \\
&= \lambda' (u', w)_G + \lambda'' (u'', w)_G \\
&= (\lambda' u' + \lambda'' u'', w)_G\,.
\end{aligned}$$

Hence, from Lemma 185 (*characterization of orthogonal projection onto subspace*), and Theorem 183 (*orthogonal projection onto complete subspace*, orthogonal projection is unique), we have

$$P_F(\lambda' u' + \lambda'' u'') = \lambda' P_F(u') + \lambda'' P_F(u'').$$

Therefore, from Lemma 92 (*linear map preserves linear combinations*), P_F is a linear map.

Continuity. Let $u \in G$.

Case $P_F(u) = 0_G$. Then, from Lemma 107 (*norm is nonnegative*, for $\|\cdot\|_G$), we have $\|P_F(u)\|_G = 0 \leq \|u\|_G$.

Case $P_F(u) \neq 0_G$. Then, from Lemma 174 (*squared norm*), Lemma 185 (*characterization of orthogonal projection onto subspace*, with $w = P_F(u) \in F$), and Lemma 175 (*Cauchy–Schwarz inequality with norms*), we have

$$\|P_F(u)\|_G^2 = (P_F(u), P_F(u))_G = (u, P_F(u))_G \leq \|u\|_G \, \|P_F(u)\|_G\,.$$

Hence, from Definition 102 (*norm*, $\|\cdot\|_G$ is definite), and **ordered field properties of $\mathbb{R}$**, we have $\|P_F(u)\|_G \leq \|u\|_G$.

Therefore, from Definition 46 (*Lipschitz continuity*), P_F is 1-Lipschitz continuous. □

Definition 187 ***(orthogonal complement).*** Let $(G, (\cdot,\cdot)_G)$ be an inner product space. Let F be a subspace of G. The *orthogonal complement of F in G*, denoted $F^\perp$, is defined by

$$F^\perp = \{u \in G \,|\, \forall v \in F,\ (u, v)_G = 0\}. \tag{113}$$

Lemma 188 ***(trivial orthogonal complements).*** *Let $(G, (\cdot,\cdot)_G)$ be an inner product space. Then, $G^\perp = \{0_G\}$ and $\{0_G\}^\perp = G$.*

Proof. From Lemma 77 (*trivial subspaces*), G and $\{0_G\}$ are subspaces of G. From Definition 187 (*orthogonal complement*), $G^\perp$ and $\{0_G\}^\perp$ are subsets of G.

Let $u \in G$ be a vector. Then, from Lemma 167 (*inner product with zero is zero*, for $(\cdot,\cdot)_G$), we have $(0_G, u)_G = (u, 0_G)_G = 0$. Hence, from Definition 187 (*orthogonal complement*), $\{0_G\}$ is a subset of $G^\perp$ and G is a subset of $\{0_G\}^\perp$.

Let $u \in G^\perp$ be a vector in the orthogonal. Let $v = u \in G$. Then, from Definition 187 (*orthogonal complement*), we have $(u, v)_G = (u, u)_G = 0$. Thus, from Definition 161 (*inner product*, $(\cdot,\cdot)_G$ is definite), we have $u = 0_G$. Hence, $G^\perp$ is a subset of $\{0_G\}$.

Therefore, $G^\perp = \{0_G\}$ and $\{0_G\}^\perp = G$. □

Lemma 189 ***(orthogonal complement is subspace).*** *Let $(G, (\cdot,\cdot)_G)$ be an inner product space. Let F be a subspace of G. Then, $F^\perp$ is a subspace of G.*

Proof. Let $v \in F$. Then, from Lemma 167 (*inner product with zero is zero*, for $(\cdot,\cdot)_G$), we have $(0_G, v)_G = 0$. Hence, from Definition 187 (*orthogonal complement*), 0_G belongs to $F^\perp$.

Let $\lambda, \lambda' \in \mathbb{R}$. Let $u, u' \in F^\perp$. Let $v \in F$. From Definition 161 (*inner product*, $(\cdot,\cdot)_G$ is a bilinear map), Definition 65 (*bilinear map*, $(\cdot,\cdot)_G$ is left linear), and **field properties of $\mathbb{R}$**, we have

$$(\lambda u + \lambda' u', v)_G = \lambda\, (u, v)_G + \lambda'\, (u', v)_G = \lambda 0 + \lambda' 0 = 0.$$

Thus, from Definition 187 (*orthogonal complement*), $\lambda u + \lambda' u'$ belongs to $F^\perp$. Hence, $F^\perp$ is closed under linear combination. Therefore, from Lemma 79 (*closed under linear combination is subspace*), $F^\perp$ is a subspace of G. □

Lemma 190 ***(zero intersection with orthogonal complement).*** *Let $(G, (\cdot,\cdot)_G)$ be an inner product space. Let F be a subspace of G. Then,*

$$F \cap F^\perp = \{0_G\}. \tag{114}$$

Proof. Let $\|\cdot\|_G$ be the norm associated with inner product $(\cdot,\cdot)_G$. Let $u \in F \cap F^\perp$. Then, $u \in F$ and $v = u \in F^\perp$. Thus, from Definition 187 (*orthogonal complement*), we have

$$(u, u)_G = (u, v)_G = 0.$$

Therefore, from Definition 161 (*inner product*, $(\cdot,\cdot)_G$ is definite), $u = 0_G$. □

Theorem 191 ***(direct sum with orthogonal complement when complete).*** *Assume hypotheses of Theorem 183 (*orthogonal projection onto complete subspace*). Then,*

$$G = F \oplus F^\perp. \tag{115}$$

Moreover, for all $u \in G$, the (unique) decomposition onto $F \oplus F^\perp$ is

$$u = P_F(u) + (u - P_F(u)) \tag{116}$$

and we have the following characterizations of the orthogonal complements:

$$u \in F \quad \Longleftrightarrow \quad P_F(u) = u; \tag{117}$$
$$u \in F^\perp \quad \Longleftrightarrow \quad P_F(u) = 0_G. \tag{118}$$

Proof. Let $u \in G$.

Then, from Definition 184 (*orthogonal projection onto complete subspace*), and Lemma 185 (*characterization of orthogonal projection onto subspace*), there exists a unique $P_F(u) \in F$ characterized by, for all $w \in F$, $(P_F(u), w)_G = (u, w)_G$. Thus, from Definition 161 (*inner product*, $\underline{(\cdot,\cdot)_G}$ is a bilinear map), Definition 65 (*bilinear map*, $\underline{(\cdot,\cdot)_G}$ is left linear), and Definition 187 (*orthogonal complement*), $u - P_F(u)$ belongs to $F^\perp$. From Definition 58 (*vector space*, $(G,+)$ is an abelian group), we have

$$u = P_F(u) + (u - P_F(u)).$$

Hence, from Definition 81 (*sum of subspaces*), $G = F + F^\perp$. Therefore, from Lemma 190 (*zero intersection with orthogonal complement*, for F), and Lemma 84 (*equivalent definitions of direct sum*), we have $G = F \oplus F^\perp$. From Definition 83 (*direct sum of subspaces*), the decomposition (116) with $P_F(u) \in F$ and $u - P_F(u) \in F^\perp$ is unique.

From Lemma 190 (*zero intersection with orthogonal complement*), 0_G belongs to both F and $F^\perp$.

(117): "left" implies "right". Assume that $u \in F$. Then, from Definition 58 (*vector space*, $(G,+)$ is an abelian group), $u = u + 0_G$ is a decomposition over $F \oplus F^\perp$. From uniqueness of the decomposition, we have $P_F(u) = u$.

(117): "right" implies "left". Conversely, assume now that $P_F(u) = u$. Then, from Definition 184 (*orthogonal projection onto complete subspace*, P_F is a mapping to F), $u = P_F(u)$ belongs to F.

(118): "left" implies "right". Assume that $u \in F^\perp$. Then, from Definition 58 (*vector space*, $(G,+)$ is an abelian group), $u = 0_G + u$ is a decomposition over $F \oplus F^\perp$. From uniqueness of the decomposition, we have $P_F(u) = 0_G$.

(118): "right" implies "left". Conversely, assume now that $P_F(u) = 0_G$. Let $v \in F$. Then, from Lemma 185 (*characterization of orthogonal projection onto subspace*), and Lemma 167 (*inner product with zero is zero*), we have

$$(u, v)_G = (P_F(u), v)_G = (0_G, v)_G = 0.$$

Hence, from Definition 187 (*orthogonal complement*), u belongs to $F^\perp$. □

Lemma 192 ***(sum is orthogonal sum).*** *Assume hypotheses of Theorem 183 (*orthogonal projection onto complete subspace*). Let $u \in G$ be a vector. Then, there exists $u' \in F^\perp$ such that $F + \mathrm{span}(\{u\}) = F + \mathrm{span}(\{u'\})$.*

Proof. Let $u' = u - P_F(u)$. Then, from Theorem 191 (*direct sum with orthogonal complement when complete*), $P_F(u)$ belongs to F and u' belongs to $F^\perp$.

Let $w \in F + \mathrm{span}(\{u\})$. Then, from Definition 81 (*sum of subspaces*), and Definition 80 (*linear span*), there exists $v \in F$ and $\lambda \in \mathbb{R}$ such that $w = v + \lambda u$. From Lemma 79 (*closed under linear* *combination is subspace*, with 1 and λ), we have $v' = v + \lambda P_F(u) \in F$, and thus, from Definition 58 (*vector space*, $(G, +)$ is an abelian group), we have

$$w = v + \lambda u = v + \lambda P_F(u) + \lambda u' = v' + \lambda u'$$

with $v' \in F$. Hence, w belongs to $F + \operatorname{span}(\{u'\})$, and thus $F + \operatorname{span}(\{u\}) \subset F + \operatorname{span}(\{u'\})$.

Let $w \in F + \operatorname{span}(\{u'\})$. Similarly, from Definition 81 (*sum of subspaces*), and Definition 80 (*linear span*), there exists $v \in F$ and $\lambda \in \mathbb{R}$ such that $w = v + \lambda u'$; from Lemma 79 (*closed under linear combination is subspace*, with 1 and $-\lambda$), we have $v' = v - \lambda P_F(u) \in F$, and thus, from Definition 58 (*vector space*, $(G, +)$ is an abelian group), we have

$$w = v + \lambda u' = v - \lambda P_F(u) + \lambda u = v' + \lambda u$$

with $v' \in F$. Hence, w belongs to $F + \operatorname{span}(\{u\})$, and thus $F + \operatorname{span}(\{u'\}) \subset F + \operatorname{span}(\{u\})$.

Therefore, $F + \operatorname{span}(\{u\}) = F + \operatorname{span}(\{u'\})$. □

Lemma 193 ***(sum of complete subspace and linear span is closed).*** *Assume hypotheses of Theorem 183 (*orthogonal projection onto complete subspace*). Let u be a nonzero vector in the orthogonal of F. Then, $F \oplus \operatorname{span}(\{u\})$ is closed for distance d_G.*

Proof. From Lemma 190 (*zero intersection with orthogonal complement*), u does not belong to F, thus from Lemma 85 (*direct sum with linear span*), the sum $F + \operatorname{span}(\{u\})$ is direct.

From Lemma 186 (*orthogonal projection is continuous linear map*, F is complete for distance d_F), P_F is a continuous linear map. Then, from Lemma 146 (*identity map is continuous*), Theorem 147 (*normed vector space of continuous linear maps*), and Lemma 79 (*closed under linear combination is subspace*), $Id - P_F$ is also a continuous linear map.

Let $(w_n)_{n\in\mathbb{N}}$ be a sequence in $F \oplus \operatorname{span}(\{u\})$. Assume that this sequence is convergent with limit $w \in G$. From Definition 81 (*sum of subspaces*), and Definition 80 (*linear span*), for all $n \in \mathbb{N}$, there exists $v_n \in F$ and $\lambda_n \in \mathbb{R}$ such that $w_n = v_n + \lambda_n u$. Then, from Lemma 44 (*compatibility of limit with continuous functions*, P_F and $Id - P_F$ are continuous), the sequences $(w'_n)_{n\in\mathbb{N}} = P_F((w_n)_{n\in\mathbb{N}})$ and $(w''_n)_{n\in\mathbb{N}} = (Id - P_F)((w_n)_{n\in\mathbb{N}})$ are also convergent, respectively with limits $w' = P_F(w)$ and $w'' = (Id - P_F)(w) = w - w'$. From Theorem 191 (*direct sum with orthogonal complement when complete*), we have, $w' \in F$ and $w'' \in F^\perp$, and for all $n \in \mathbb{N}$,

$$w''_n = (Id - P_F)(w_n) = (Id - P_F)(v_n + \lambda_n u) = v_n + \lambda_n u - v_n = \lambda_n u.$$

Thus, $(w''_n)_{n\in\mathbb{N}}$ is also a sequence of $\operatorname{span}(\{u\})$. Then, from, Lemma 112 (*linear span is closed*, $\operatorname{span}(\{u\})$ is closed), and Lemma 31 (*closed is limit of sequences*), the limit w'' actually belongs to $\operatorname{span}(\{u\})$. Hence, from Definition 80 (*linear span*), there exists $\lambda \in \mathbb{R}$ such that $w'' = \lambda u$. And we have

$$w = w' + w'' = w' + \lambda u \in F \oplus \operatorname{span}(\{u\}).$$

Therefore, from Lemma 31 (*closed is limit of sequences*), $F \oplus \operatorname{span}(\{u\})$ is closed for distance d_G. □

4.6 Hilbert space

Definition 194 ***(Hilbert space).*** Let $(H, (\cdot,\cdot)_H)$ be an inner product space. Let $\|\cdot\|_H$ be the norm associated with inner product $(\cdot,\cdot)_H$ through Definition 172 (*square root of inner square*), and Lemma 177 (*inner product gives norm*). Let d_H be the distance associated with norm $\|\cdot\|_H$ through Lemma 111 (*norm gives distance*). $(H, (\cdot,\cdot)_H)$ is an *Hilbert space* iff (H, d_H) is a complete metric space.

Lemma 195 ***(closed Hilbert subspace).*** *Let $(H, (\cdot,\cdot)_H)$ be a Hilbert space. Let H_h be a closed subspace of H. Then, H_h equipped with the restriction to H_h of the inner product $(\cdot,\cdot)_H$ is a Hilbert space.*

Proof. Direct consequence of Lemma 166 (*inner product subspace*, H_h is a subspace of H), Definition 74 (*subspace*, H_h is a subset of H), Lemma 194 (*Hilbert space*, H is complete), Lemma 39 (*closed subset of complete is complete*, F is closed), and Definition 194 (*Hilbert space*). □

Theorem 196 ***(Riesz–Fréchet).*** *Let* $(H, (\cdot,\cdot)_H)$ *be a Hilbert space. Let* $\|\cdot\|_H$ *be the norm associated with inner product* $(\cdot,\cdot)_H$*. Let* $\varphi \in H'$ *be a continuous linear form on* H*. Then, there exists a unique vector* $u \in H$ *such that*

$$\forall v \in H, \quad \langle \varphi | v \rangle_{H',H} = (u, v)_H \,. \tag{119}$$

Moreover, the mapping $\tau : H' \to H$ *defined by*

$$\forall \varphi \in H', \quad \tau(\varphi) = u, \tag{120}$$

where u *is characterized by* (119)*, is a continuous isometric isomorphism from* H' *onto* H*.*

Proof. From Definition 194 (*Hilbert space*, $(H, (\cdot,\cdot)_H)$ is an inner product space), and Definition 165 (*inner product space*), H is a space.

Uniqueness. Let $u, u' \in H$ be two vectors such that

$$\forall v \in H, \quad \langle \varphi | v \rangle_{H',H} = \varphi(v) = (u, v)_H = (u', v)_H \,.$$

Let $v \in H$ be a vector. Then, from Definition 161 (*inner product*, $(\cdot,\cdot)_H$ is a bilinear map), Definition 65 (*bilinear map*, $(\cdot,\cdot)_H$ is left linear), and Definition 70 (*vector subtraction*), we have $(u - u', v)_H = 0$. Thus, from Definition 187 (*orthogonal complement*), and Lemma 188 (*trivial orthogonal complements*), $u - u'$ belongs to $H^\perp = \{0_H\}$. Hence, from Definition 58 (*vector space*, $(H, +)$ is an abelian group), $u = u'$.

Existence.

Case $\boldsymbol{\varphi = 0_{H'}}$. Then, from Definition 58 (*vector space*, 0_H belongs to H), let $u = 0_H$ be the zero vector. Let $v \in H$ be a vector. Then, from Lemma 188 (*trivial orthogonal complements*, $H^\perp = \{0_H\}$), we have

$$\langle \varphi | v \rangle_{H',H} = \varphi(v) = 0_{H'}(v) = 0 = (0_H, v)_H = (u, v)_H \,.$$

Case $\boldsymbol{\varphi \neq 0_{H'}}$. Then, let $u_0 \in H$ such that $\varphi(u_0) \neq 0$. Let F be the kernel of φ. Then, from Definition 98 (*kernel*), $u_0 \notin F$. Moreover, from Lemma 149 (*continuous linear maps have closed kernel*, for φ), and Lemma 99 (*kernel is subspace*), F is a closed subspace of H. Thus, from Lemma 195 (*closed Hilbert subspace*), F is a complete subspace of H. Hence, from Theorem 183 (*orthogonal projection onto complete subspace*), and Definition 184 (*orthogonal projection onto complete subspace*), let P_F be the orthogonal projection onto F. Then, from Theorem 191 (*direct sum with orthogonal complement when complete*, decomposition and contrapositive of (117)), we have

$$P_F(u_0) \in F, \quad u_0 - P_F(u_0) \in F^\perp \quad \text{and} \quad P_F(u_0) \neq u_0.$$

Thus, from Definition 98 (*kernel*, $F = \ker(\varphi)$), Definition 64 (*linear form*, φ is a linear map), Definition 62 (*linear map*, φ is additive), and Definition 70 (*vector subtraction*), we have

$$\varphi(P_F(u_0)) = 0 \quad \text{and} \quad \varphi(u_0 - P_F(u_0)) = \varphi(u_0).$$

Let $v_0 = u_0 - P_F(u_0)$. Then,

$$v_0 \in F^\perp \quad \text{and} \quad \varphi(v_0) = \varphi(u_0) \neq 0.$$

Moreover, from Definition 58 (*vector space*, $(H, +)$ is an abelian group), and Definition 70 (*vector subtraction*), we have $v_0 \neq 0_H$. Thus, from Definition 102 (*norm*, $\|\cdot\|_H$ is definite, contrapositive), we have $\|v_0\|_H \neq 0$.

Let $\xi_0 = \frac{v_0}{\|v_0\|_H}$. Then, from Lemma 189 (*orthogonal complement is subspace*, $F^\perp$ is subspace), Lemma 78 (*closed under vector operations is subspace*, $F^\perp$ is closed under scalar multiplication), Definition 71 (*scalar division*), Definition 64 (*linear form*, φ is a linear map), Definition 62 (*linear map*, φ is homogeneous of degree 1), Definition 71 (*scalar division*), Lemma 73 (*zero-product property*, contrapositive), and **field properties of $\mathbb{R}$**, we have

$$\xi_0 \in F^\perp, \quad \varphi(\xi_0) = \frac{\varphi(v_0)}{\|v_0\|_H} \neq 0 \quad \text{and} \quad \xi_0 \neq 0_H.$$

Moreover, from Lemma 108 (*normalization by nonzero*, with $\lambda = 1$), and **field properties of $\mathbb{R}$**, we have $\|\xi_0\|_H^2 = 1$.

Let $u = \varphi(\xi_0)\xi_0$. Then, from Lemma 189 (*orthogonal complement is subspace*, $F^\perp$ is subspace), and Lemma 78 (*closed under vector operations is subspace*, $F^\perp$ is closed under scalar multiplication), $u \in F^\perp$.

Let $v \in H$ be a vector. Since $\varphi(\xi_0) \neq 0$, let $\lambda = \frac{\varphi(v)}{\varphi(\xi_0)}$ and $w = v - \lambda\xi_0$. Then, from Definition 64 (*linear form*, φ is a linear map), Lemma 92 (*linear map preserves linear combinations*), Definition 70 (*vector subtraction*), and Definition 71 (*scalar division*), we have

$$\varphi(w) = \varphi(v) - \lambda\varphi(\xi_0) = \varphi(v) - \frac{\varphi(v)}{\varphi(\xi_0)}\varphi(\xi_0) = 0.$$

Thus, from Definition 98 (*kernel*, $F = \ker(\varphi)$), w belongs to F. Hence, from **field properties of $\mathbb{R}$ (with $\varphi(\xi_0) \neq 0$)**, Lemma 174 (*squared norm*, $\|\xi_0\|_H^2 = 1$), Definition 161 (*inner product*, $(\cdot,\cdot)_H$ is a bilinear map), Definition 65 (*bilinear map*, $(\cdot,\cdot)_H$ is left linear), and Definition 187 (*orthogonal complement*, $u \in F^\perp$ and $w \in F$), we have

$$\begin{aligned}
(u,v)_H - \varphi(v) &= (u,v)_H - \varphi(v)\frac{\varphi(\xi_0)}{\varphi(\xi_0)}(\xi_0,\xi_0)_H \\
&= (u,v)_H - \lambda\,(u,\xi_0)_H \\
&= (u, v - \lambda\xi_0)_H \\
&= (u,w)_H \\
&= 0.
\end{aligned}$$

Hence, from Definition 155 (*bra-ket notation*), and **field properties of $\mathbb{R}$**, we have

$$\langle\varphi|v\rangle_{H',H} = \varphi(v) = (u,v)_H\,.$$

Linearity. From Lemma 154 (*topological dual is complete normed vector space*, for H), and Definition 104 (*normed vector space*), H' is a space.

Let $\tau : H' \to H$ be the mapping defined by, for all $\varphi \in H'$, $\tau(\varphi) = u$ where u is uniquely characterized by

$$\forall v \in H, \quad \langle\varphi|v\rangle_{H',H} = \varphi(v) = (u,v)_H\,. \tag{121}$$

Let $\lambda', \lambda'' \in \mathbb{K}$ be scalars. Let $\varphi', \varphi'' \in H'$ be continuous linear forms on H. Then, $\tau(\varphi')$ and $\tau(\varphi'')$ belong to H. Thus, from Definition 58 (*vector space*, H' and H are closed under vector operations), $\varphi = \lambda'\varphi' + \lambda''\varphi''$ belongs to H' and $u = \lambda'\tau(\varphi') + \lambda''\tau(\varphi'')$ belongs to H. Let $v \in H$ be a vector. Then, from Lemma 156 (*bra-ket is bilinear map*), Definition 161 (*inner product*, $(\cdot,\cdot)_H$ is a bilinear map), Definition 65 (*bilinear map*, bra-ket and $(\cdot,\cdot)_H$ are left linear), and characterization (121), we have

$$\begin{aligned}
\langle\varphi|v\rangle_{H',H} &= \langle\lambda'\varphi' + \lambda''\varphi''|v\rangle_{H',H} \\
&= \lambda'\,\langle\varphi'|v\rangle_{H',H} + \lambda''\,\langle\varphi''|v\rangle_{H',H} \\
&= \lambda'\,(\tau(\varphi'),v)_H + \lambda''\,(\tau(\varphi''),v)_H \\
&= (\lambda'\tau(\varphi') + \lambda''\tau(\varphi''),v)_H \\
&= (u,v)_H\,.
\end{aligned}$$

Hence, from unique characterization (121), we have

$$\tau(\lambda'\varphi' + \lambda''\varphi'') = \tau(\varphi) = u = \lambda'\tau(\varphi') + \lambda''\tau(\varphi'').$$

Therefore, from Definition 62 (*linear map*), τ is a linear map from H' to H.

Isomorphism. Let $\varphi \in H'$ be a continuous linear form on H. Assume that $\tau(\varphi) = 0_H$. Let $v \in H$ be a vector. Then, from characterization (121), and Lemma 167 (*inner product with zero is zero*), we have

$$\varphi(v) = (\tau(\varphi), v)_H = (0_H, v)_H = 0.$$

Thus, $\varphi = 0_{H'}$ is the zero linear form. Hence, from Definition 98 (*kernel*, $\underline{\ker(\tau) = \{0_{H'}\}}$), and Lemma 100 (*injective linear map has zero kernel*), τ is injective.

Let $u \in H$ be a vector. Let $\varphi : H \to \mathbb{K}$ be the mapping defined by, for all $v \in H$, $\varphi(v) = (u, v)_H$. Then, from Definition 161 (*inner product*, $\underline{(\cdot,\cdot)_H \text{ is a bilinear map}}$), Definition 65 (*bilinear map*, $\underline{(\cdot,\cdot)_H \text{ is right linear}}$), and Definition 64 (*linear form*), φ is a linear form on H. Let $v \in H$ be a vector. Then, from Lemma 175 (*Cauchy–Schwarz inequality with norms*), we have

$$|\varphi(v)| = |\,(u, v)_H\,| \leq \|u\|_H \; \|v\|_H \,.$$

Thus, from Definition 139 (*bounded linear map*, $\underline{\text{with } C = \|u\|_H \geq 0}$), and Theorem 142 (*continuous linear map*, $\underline{5 \Rightarrow 2}$), φ is continuous. Hence, from Definition 152 (*topological dual*), φ belongs to H'. Moreover, from characterization (121), we have $\tau(\varphi) = u$. Hence, from **the definition of a surjective function**, τ is surjective.

Therefore, from **the definition of a bijective function**, τ is bijective, and from Definition 97 (*isomorphism*), τ is an isomorphism from H' onto H.

Isometry. Let $\varphi \in H'$ be a continuous linear form on H. Let $u = \tau(\varphi) \in H$.

Case $\varphi = 0_{H'}$. Then, from Lemma 91 (*linear map preserves zero*, $\underline{\tau \text{ is a linear map}}$), we have $u = 0_H$. Hence, from Lemma 106 (*norm preserves zero*, $\underline{\text{for } \|\cdot\|_{H'} \text{ and } \|\cdot\|_H}$), we have

$$\|\tau(\varphi)\|_H = \|u\|_H = 0 = \|\varphi\|_{H'} \,.$$

Case $\varphi \neq 0_{H'}$. Then, from Definition 98 (*kernel*, $\underline{\ker(\varphi) = \{0_{H'}\}}$), we have $u \neq 0_H$. Thus, from Definition 102 (*norm*, $\underline{\|\cdot\|_H \text{ is definite, contrapositive}}$), $\|u\|_H \neq 0$. Hence, from characterization (121) (with $v = u$), Lemma 174 (*squared norm*, $\underline{\text{for } \|\cdot\|_H}$), **nonnegativeness of the square function in $\mathbb{R}$**, and **field properties of $\mathbb{R}$ (with $\|u\|_H \neq 0$)**, we have

$$\frac{|\varphi(u)|}{\|u\|_H} = \frac{|\,(u, u)_H\,|}{\|u\|_H} = \frac{\|u\|_H^2}{\|u\|_H} = \|u\|_H \,.$$

Hence, from Definition 153 (*dual norm*), Definition 135 (*operator norm*), and Definition 2 (*supremum*, $\underline{\|\varphi\|_{H'} \text{ is an upper bound for } \left\{ \frac{|\varphi(v)|}{\|v\|_H} \,\middle|\, v \in H,\, v \neq 0_H \right\}}$), we have

$$\|u\|_H \leq \|\varphi\|_{H'} \,.$$

Finally, let $v \in H$ be a vector. Assume that $v \neq 0_H$. Then, from Definition 102 (*norm*, $\underline{\|\cdot\|_H \text{ is definite, contrapositive}}$), Lemma 107 (*norm is nonnegative*, $\underline{\text{for } \|\cdot\|_H}$), Lemma 175 (*Cauchy–Schwarz inequality with norms*), and **ordered field properties of $\mathbb{R}$ (with $\|v\|_H > 0$)**, we have

$$\frac{|\varphi(v)|}{\|v\|_H} = \frac{|\,(u, v)_H\,|}{\|v\|_H} \leq \|u\|_H \,.$$

Thus, from Lemma 144 (*finite operator norm is continuous*, $\underline{\|u\|_H \text{ is an upper bound for the subset } \left\{ \frac{|\varphi(v)|}{\|v\|_H} \,\middle|\, v \in H,\, v \neq 0_H \right\}}$), and Definition 153 (*dual norm*), we have

$$\|\varphi\|_{H'} \leq \|u\|_H \,.$$

Hence, $\|\tau(\varphi)\|_H = \|u\|_H = \|\varphi\|_{H'}$.

Therefore, from Definition 122 (*linear isometry*), τ is a linear isometry from H' to H.

Continuity. From Lemma 145 (*linear isometry is continuous*, τ is a linear isometry from H' to H), τ belongs to $\mathcal{L}_c(H', H)$. □

Lemma 197 ***(compatible ρ for Lax–Milgram).*** *Let $\alpha, C \in \mathbb{R}$. Assume that $0 < \alpha \leq C$. Then,*

$$\forall \rho \in \mathbb{R}, \quad 0 < \rho < \frac{2\alpha}{C^2} \quad \Longrightarrow \quad 0 \leq \sqrt{1 - 2\rho\alpha + \rho^2 C^2} < 1. \tag{122}$$

Proof. From hypothesis ($0 < \alpha \leq C$), **ordered field properties of $\mathbb{R}$**, and **increase of the square function over $\mathbb{R}^+$**, we have $0 < \frac{\alpha^2}{C^2} \leq 1$. Let $\rho \in \mathbb{R}$. Then, from **field properties of $\mathbb{R}$**, we have

$$1 - 2\rho\alpha + \rho^2 C^2 = \left(\rho C - \frac{\alpha}{C}\right)^2 + 1 - \frac{\alpha^2}{C^2} \geq 0.$$

Assume that $0 < \rho < \frac{2\alpha}{C^2}$. Then, from **ordered field properties of $\mathbb{R}$ (with $C > 0$ and $\rho > 0$)**, we successively have $\rho C^2 < 2\alpha$, $\rho^2 C^2 < 2\rho\alpha$ and $1 - 2\rho\alpha + \rho^2 C^2 < 1$. Hence, from **compatibility of the square root with comparison in $\mathbb{R}^+$**, we have

$$0 = \sqrt{0} \leq \sqrt{1 - 2\rho\alpha + \rho^2 C^2} < \sqrt{1} = 1.$$

□

Theorem 198 ***(Lax–Milgram).*** *Let $(H, (\cdot,\cdot)_H)$ be a real Hilbert space. Let $\|\cdot\|_H$ be the norm associated with inner product $(\cdot,\cdot)_H$. Let H' be the topological dual of H. Let $\|\cdot\|_{H'}$ be the dual norm associated with $\|\cdot\|_H$. Let a be a bounded bilinear form on H. Let $f \in H'$ be a continuous linear form on H. Assume that a is coercive with constant $\alpha > 0$. Then, there exists a unique $u \in H$ solution to Problem (2). Moreover,*

$$\|u\|_H \leq \frac{1}{\alpha} \|f\|_{H'}. \tag{123}$$

Proof. Let d_H be the distance associated with norm $\|\cdot\|_H$. Then, from Definition 194 (*Hilbert space*), $(H, (\cdot,\cdot)_H)$ is an inner product space and (H, d_H) is a complete metric space. Thus, from Lemma 177 (*inner product gives norm*), $(H, \|\cdot\|_H)$ is a normed vector space. Moreover, from Lemma 154 (*topological dual is complete normed vector space*), $(H', \|\cdot\|_{H'})$ is a also normed vector space. Hence, from Definition 104 (*normed vector space*), H and H' are both spaces.

Existence and uniqueness. From Lemma 158 (*representation for bounded bilinear form*, for a), let $A \in \mathcal{L}_c(H, H')$ be the (unique) continuous linear map from H to H' such that

$$\forall u, v \in H, \quad a(u, v) = \langle A(u)|v\rangle_{H',H}.$$

Then, from Definition 159 (*coercive bilinear form*, for a), we have

$$\forall u \in H, \quad \langle A(u)|u\rangle_{H',H} = a(u, u) \geq \alpha \|u\|_H^2. \tag{124}$$

From Definition 157 (*bounded bilinear form*), let $C \geq 0$ be a continuity constant of a. Then, from Lemma 148 (*operator norm estimation*, in $\mathcal{L}_c(H, H')$), and Lemma 158 (*representation for bounded bilinear form*, for a), we have

$$\forall u \in H, \quad \|A(u)\|_{H'} \leq |||A|||_{H',H} \|u\|_H \leq C \|u\|_H. \tag{125}$$

Let $u \in H$ be a vector. Then, from Theorem 196 (*Riesz–Fréchet*, for $\varphi = A(u)$ and $\varphi = f$), $\tau(A(u)), \tau(f) \in H$ are the (unique) vectors such that

$$\forall v \in H, \quad a(u, v) = \langle A(u)|v\rangle_{H',H} = (\tau(A(u)), v)_H;$$
$$\forall v \in H, \quad f(v) = \langle f|v\rangle_{H',H} = (\tau(f), v)_H.$$

Moreover, from (124), **ordered field properties of $\mathbb{R}$**, (125), and Definition 122 (*linear isometry*, $\underline{\tau \text{ is a linear isometry}}$), we have

$$\forall u \in H, \quad -(\tau(A(u)), u)_H = -\langle A(u)|u\rangle_{H',H} \leq -\alpha \|u\|_H^2 \,; \tag{126}$$

$$\forall u \in H, \quad \|\tau(A(u))\|_H = \|A(u)\|_{H'} \leq C \|u\|_H \,. \tag{127}$$

Let $u, v \in H$ be vectors. Then, from Definition 161 (*inner product*, $\underline{(\cdot,\cdot)_H \text{ is a bilinear map}}$), and Definition 65 (*bilinear map*, $\underline{(\cdot,\cdot)_H \text{ is left linear}}$), we have the equivalences

$$\begin{aligned} a(u,v) = f(v) \quad &\Leftrightarrow \quad (\tau(A(u)), v)_H = (\tau(f), v)_H \\ &\Leftrightarrow \quad (\tau(A(u)) - \tau(f), v)_H = 0. \end{aligned}$$

Hence, from Definition 187 (*orthogonal complement*, $\underline{\tau(A(u)) - \tau(f) \text{ belongs to } H^\perp}$), Lemma 188 (*trivial orthogonal complements*, $\underline{H^\perp = \{0_H\}}$), and Definition 58 (*vector space*, $\underline{(H,+) \text{ is an abelian group}}$), we have the equivalence

$$\textit{Problem } (2) \quad \Longleftrightarrow \quad \text{find } u \in H \text{ such that:} \quad \tau(A(u)) = \tau(f). \tag{128}$$

From Lemma 150 (*compatibility of composition with continuity*, $\underline{\tau \text{ belongs to } \mathcal{L}_c(H',H)}$), $\tau \circ A$ belongs to $\mathcal{L}_c(H,H)$. From Lemma 160 (*coercivity constant is less than continuity constant*), we have $0 < \alpha \leq C$, hence $\frac{2\alpha}{C^2} > 0$. Let $\rho \in \mathbb{R}$ be a number. Assume that $0 < \rho < \frac{2\alpha}{C^2}$. Then, from Theorem 147 (*normed vector space of continuous linear maps*, $\underline{(\mathcal{L}_c(H,H), \|\!|\cdot|\!\|_{H,H}) \text{ is a normed vector space}}$), Definition 104 (*normed vector space*, $\underline{\mathcal{L}_c(H,H) \text{ is a space}}$), Definition 58 (*vector space*, $\underline{\mathcal{L}_c(H,H) \text{ is closed under vector operations}}$), Definition 70 (*vector subtraction*), and Lemma 146 (*identity map is continuous*), $g_0 = \mathrm{Id}_H - \rho\tau \circ A$ belongs to $\mathcal{L}_c(H,H)$.

From Definition 58 (*vector space*, $\underline{H \text{ is closed under vector operations and } \tau(f) \in H}$), let $g : H \to H$ be the mapping defined by

$$\forall v \in H, \quad g(v) = g_0(v) + \rho\tau(f).$$

Let $u \in H$ be a vector. Then, from the definition of mappings g and g_0, Definition 58 (*vector space*, $\underline{(H,+) \text{ is an abelian group and scalar multiplication is distributive wrt vector addition}}$), and Lemma 73 (*zero-product property*, $\underline{\text{with } \lambda = \rho \neq 0}$), we have

$$\begin{aligned} g(u) = u \quad &\Leftrightarrow \quad g_0(u) + \rho\tau(f) = u \\ &\Leftrightarrow \quad u - \rho\tau(A(u)) + \rho\tau(f) = u \\ &\Leftrightarrow \quad \rho(\tau(A(u)) - \tau(f)) = 0_H \\ &\Leftrightarrow \quad \tau(A(u)) = \tau(f). \end{aligned}$$

Hence, from (128), we have the equivalence

$$\textit{Problem } (2) \quad \Longleftrightarrow \quad \text{find } u \in H \text{ such that:} \quad g(u) = u. \tag{129}$$

Let $v, v' \in H$ be vectors. Then, from Definition 58 (*vector space*, $\underline{(H,+) \text{ is an abelian group}}$), and Definition 70 (*vector subtraction*), let $z = v - v' \in H$. Then, from Definition 70 (*vector subtraction*), Lemma 69 (*minus times yields opposite vector*, $\underline{\text{with } \lambda = 1}$), and Definition 58 (*vector space*, $\underline{(H,+) \text{ is an abelian group and scalar multiplication is distributive wrt vector addition}}$), we have

$$g(v) - g(v') = g_0(v) + \rho\tau(f) - (g_0(v') + \rho\tau(f)) = g_0(v - v') = g_0(z).$$

Thus, from Lemma 168 (*square expansion plus*, $\underline{\text{for } \|\cdot\|_H}$), Definition 161 (*inner product*, $\underline{(\cdot,\cdot)_H \text{ is a symmetric bilinear map}}$), Definition 65 (*bilinear map*, $\underline{(\cdot,\cdot)_H \text{ is right linear}}$), Definition 102 (*norm*, $\underline{\|\cdot\|_H \text{ is absolutely homogeneous of degree 1}}$), **ordered field properties of $\mathbb{R}$**, (126), and

(127), we have

$$
\begin{aligned}
\|g(v)-g(v')\|_H^2 &= \|g_0(z)\|_H^2 \\
&= \|z-\rho\tau(A(z))\|_H^2 \\
&= \|z\|_H^2 - 2\rho\,(\tau(A(z)),z)_H + \rho^2\,\|\tau(A(z))\|_H^2 \\
&\leq \|z\|_H^2 - 2\rho\alpha\,\|z\|_H^2 + \rho^2C^2\,\|z\|_H^2 \\
&= (1-2\rho\alpha+\rho^2C^2)\,\|v-v'\|_H^2 .
\end{aligned}
$$

Hence, from **compatibility of the square root function with comparison in $\mathbb{R}^+$**, Definition 46 (*Lipschitz continuity*, with $k=\sqrt{1-2\rho\alpha+\rho^2C^2}$), Lemma 197 (*compatible ρ for Lax–Milgram*, since $0<\alpha\leq C$ and $0<\rho<\frac{2\alpha}{C^2}$), and Definition 48 (*contraction*, since $0\leq\sqrt{1-2\rho\alpha+\rho^2C^2}<1$), g is a contraction. Then, from Theorem 56 (*fixed point*, for g contraction in (H,d_H) complete), there exists a unique fixed point $u\in H$ such that $g(u)=u$. Hence, from (129), there exists a unique solution to Problem (2).

Estimation. Let $u\in H$ be the solution to Problem (2).

Case $u=0_H$. Then, from Lemma 106 (*norm preserves zero*, for $\|\cdot\|_H$), Lemma 107 (*norm is nonnegative*, for $\|\cdot\|_{H'}$), and **ordered field properties of $\mathbb{R}$ (with $\alpha>0$)**, we have

$$\|u\|_H = 0 \leq \frac{1}{\alpha}\,\|f\|_{H'} .$$

Case $u\neq 0_H$. Then, from Definition 102 (*norm*, $\|\cdot\|_H$ is definite, contrapositive), and Lemma 107 (*norm is nonnegative*, for $\|\cdot\|_H$), we have $\|u\|_H>0$. Moreover, from Definition 159 (*coercive bilinear form*, for a), **properties of the absolute value on $\mathbb{R}$**, (2) with $v=u$, and Lemma 148 (*operator norm estimation*, for $f\in H'$), we have

$$\alpha\,\|u\|_H^2 \leq a(u,u) \leq |a(u,u)| = |f(u)| \leq \|f\|_{H'}\,\|u\|_H .$$

Hence, from **ordered field properties of $\mathbb{R}$ (with $\|u\|_H,\alpha>0$)**, we have the estimation

$$\|u\|_H \leq \frac{1}{\alpha}\,\|f\|_{H'} .$$

□

Lemma 199 *(Galerkin orthogonality).* *Let $(H,(\cdot,\cdot)_H)$ be a real Hilbert space. Let a be a bounded bilinear form on H. Let $f\in H'$ be a continuous linear form on H. Let H_h be a subspace of H. Let $u\in H$ be a solution to Problem (2). Let $u_h\in H_h$ be a solution to Problem (3). Then,*

$$\forall v_h\in H_h,\quad a(u-u_h,v_h)=0. \tag{130}$$

Proof. Let $v_h\in H_h$ be a vector. Then, from Definition 74 (*subspace*, H_h is a subset of H), v_h also belongs to H. Hence, from (2) with $v=v_h$, (3), Definition 65 (*bilinear map*, a is left linear), and **field properties of $\mathbb{R}$**, we have

$$a(u-u_h,v_h)=a(u,v_h)-a(u_h,v_h)=f(v_h)-f(v_h)=0.$$

□

Theorem 200 *(Lax–Milgram, closed subspace).* *Assume hypotheses of Theorem 198 (*Lax–Milgram*). Let H_h be a closed subspace of H. Then, there exists a unique $u_h\in H_h$ solution to Problem (3). Moreover,*

$$\|u_h\|_H \leq \frac{1}{\alpha}\,\|f\|_{H'} . \tag{131}$$

Proof. Direct consequence of Lemma 195 (*closed Hilbert subspace*, H_h is a closed subspace of H), and Theorem 198 (*Lax–Milgram*, $(H_h, (\cdot,\cdot)_H)$ is a Hilbert space) where the restriction to H_h of the norm associated to $(\cdot,\cdot)_H$ is still denoted $\|\cdot\|_H$. □

Lemma 201 ***(Céa).*** *Assume hypotheses of Theorem 200 (*Lax–Milgram, closed subspace*). Let $C \geq 0$ be a continuity constant of the bounded bilinear form a. Let $u \in H$ be the unique solution to Problem* (2)*. Let $u_h \in H_h$ be the unique solution to Problem* (3)*. Then,*

$$\forall v_h \in H_h, \quad \|u - u_h\|_H \leq \frac{C}{\alpha} \|u - v_h\|_H . \tag{132}$$

Proof. Let $v_h \in H_h$ be a vector in the subspace.

Case $u = u_h$. Then, from Definition 58 (*vector space*, $(H,+)$ is an abelian group), Lemma 106 (*norm preserves zero*, $u - u_h = 0_H$), Lemma 107 (*norm is nonnegative*, for $\|\cdot\|_H$), and **ordered field properties of $\mathbb{R}$ with $\alpha > 0$ and $C \geq 0$**, we have

$$\|u - u_h\|_H = 0 \leq \frac{C}{\alpha} \|u - v_h\|_H .$$

Case $u \neq u_h$. Then, from Definition 58 (*vector space*, $(H,+)$ is an abelian group), and Definition 102 (*norm*, $\|\cdot\|_H$ is definite, contrapositive), we have $\|u - u_h\|_H \neq 0$. Moreover, from Definition 70 (*vector subtraction*), Definition 161 (*inner product*, $(\cdot,\cdot)_H$ is a bilinear map), Definition 65 (*bilinear map*, $(\cdot,\cdot)_H$ is right linear), and Lemma 199 (*Galerkin orthogonality*), we have

$$a(u - u_h, u - v_h) = a(u - u_h, u) - a(u - u_h, v_h) = a(u - u_h, u). \tag{133}$$

Thus, from Definition 159 (*coercive bilinear form*, for a with $u = u - u_h$), **properties of the absolute value on $\mathbb{R}$**, (133) with u_h and v_h in H_h, compatibility of the absolute value with comparison in $\mathbb{R}$, and Definition 157 (*bounded bilinear form*), we have

$$\begin{array}{rcl} \alpha \|u - u_h\|_H^2 & \leq & a(u - u_h, u - u_h) \\ & \leq & |a(u - u_h, u - u_h)| \\ & = & |a(u - u_h, u)| \\ & = & |a(u - u_h, u - v_h)| \\ & \leq & C \|u - u_h\|_H \|u - v_h\|_H . \end{array}$$

Hence, from **ordered field properties of $\mathbb{R}$ with $\alpha, \|u - u_h\|_H > 0$**, we have

$$\|u - u_h\|_H \leq \frac{C}{\alpha} \|u - v_h\|_H .$$

□

Lemma 202 ***(finite dimensional subspace in Hilbert space is closed).*** *Let $(H, (\cdot,\cdot)_G)$ be a real Hilbert space. Let $\|\cdot\|_H$ be the norm associated with inner product $(\cdot,\cdot)_H$. Let d_H be the distance associated with norm $\|\cdot\|_H$. Let F be a subspace of H. Assume that F is a finite dimensional subspace. Then, F is closed for distance d_H.*

Proof. From Definition 82 (*finite dimensional subspace*), let $n \in \mathbb{N}$, and let $u_1, \ldots, u_n \in H$ such that $F = \mathrm{span}(\{u_1, \ldots, u_n\}) = \mathrm{span}(\{u_1\}) + \ldots + \mathrm{span}(\{u_n\})$. For $i \in \mathbb{N}$ with $1 \leq i \leq n$, let $F_i = \mathrm{span}(\{u_1, \ldots, u_i\})$. Then, for $2 \leq i \leq n$, we have $F_i = F_{i-1} + \mathrm{span}(\{u_i\})$. Let $P(i)$ be the property "F_i is closed for distance d_H".

Induction: $P(1)$. From Lemma 112 (*linear span is closed*), $F_1 = \mathrm{span}(\{u_1\})$ is closed for distance d_H.

Induction: $P(i-1)$ implies $P(i)$. Assume that $2 \leq i \leq n$. Assume that $P(i-1)$ holds. Then, from Lemma 192 (*sum is orthogonal sum*), there exists $u'_i \in F_{i-1}^{\perp}$ such that

$$F_i = F_{i-1} + \mathrm{span}(\{u_i\}) = F_{i-1} + \mathrm{span}(\{u'_i\}).$$

Case $u'_i = 0_G$. Then, $F_i = F_{i-1}$ is closed for distance d_G. **Case $u'_i \neq 0_G$.** Then, from Definition 194 (*Hilbert space,* $\underline{H \text{ is complete for distance } d_H}$), and Lemma 39 (*closed subset of complete is complete*), F_{i-1} is complete for distance d_H. Thus, from Lemma 193 (*sum of complete subspace and linear span is closed*), $F_i = F_{i-1} + \mathrm{span}(\{u'_i\})$ is closed for distance d_G.

Hence, by (finite) induction on $i \in \mathbb{N}$ with $1 \leq i \leq n$, we have $P(n)$. Therefore, $F = F_n$ is closed for distance d_G. □

Theorem 203 ***(Lax–Milgram–Céa, finite dimensional subspace).*** *Assume hypotheses of Theorem 198 (*Lax–Milgram*). Let $C \geq 0$ be a continuity constant of the bounded bilinear form a. Let $u \in H$ be the unique solution to Problem* (2)*. Let H_h be a finite dimensional subspace of H. Then, there exists a unique $u_h \in H_h$ solution to Problem* (3)*. Moreover,*

$$\|u_h\|_H \leq \frac{1}{\alpha} \|f\|_{H'} ; \tag{134}$$

$$\forall v_h \in H_h, \qquad \|u - u_h\|_H \leq \frac{C}{\alpha} \|u - v_h\|_H . \tag{135}$$

Proof. Direct consequence of Lemma 202 (*finite dimensional subspace in Hilbert space is closed*), Theorem 200 (*Lax–Milgram, closed subspace*), and Lemma 201 (*Céa*). □

5 Conclusions, perspectives

We have presented a very detailed proof of the Lax–Milgram theorem for the resolution on a Hilbert space of linear (partial differential) equations set under their weak form. Among the various proofs available in the literature, we have chosen a path using basic notions. In particular, we have avoided to obtain the result from a more general one, e.g. set on a Banach space. The proof uses the following main arguments: the representation lemma for bounded bilinear forms, the Riesz–Fréchet representation theorem, the orthogonal projection theorem for a complete subspace, and the fixed point theorem for a contraction on a complete metric space.

The short-term purpose of this work was to help the formalization of such a result in the Coq formal proof assistant. This was recently achieved [3]. One of the key issues for the computer scientists that formalize the pen-and-paper proof was to deal with the embedded algebraic structures: group, vector space (an external operation is added), normed vector space (a norm is added), inner vector space (an inner product is added), Hilbert space (completeness is added). New Coq structures should be extensions of the previous ones: the addition operation in the Hilbert space should be the very same addition operation from the initial group structure.

The long-term purpose of these studies is the formal proof of programs implementing the Finite Element Method. For instance, considering the standard Laplace equation (1), one proves that it can be written in weak formulation as

$$\text{find } u \in H_0^1(\Omega) \text{ such that:} \quad \forall v \in H_0^1(\Omega), \int_\Omega \nabla u \cdot \nabla v = \int_\Omega fv, \tag{136}$$

where $H_0^1(\Omega)$ is a Sobolev space. Problem (136) takes the form of Problem (2), with the following notations: $H = H_0^1(\Omega)$, the bilinear form is defined by $a(v,w) = \int_\Omega \nabla v \cdot \nabla w$, and the linear form by $f(v) = \int_\Omega qv$. To apply the Lax–Milgram theorem, one needs to prove in particular that $H_0^1(\Omega)$ is a Hilbert space.

As a consequence, we will have to write very detailed pen-and-paper proofs for the following notions and results: large parts of the integration and distribution theories, define Sobolev spaces (at least $L^2(\Omega)$, $H^1(\Omega)$ and $H_0^1(\Omega)$ for some bounded domain Ω of $\mathbb{R}^d$ with $d = 1$, 2, or 3), and prove that they are Hilbert spaces. And finally, many results of the interpolation and approximation theory to define the Finite Element Method itself.

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A Lists of statements

List of Definitions

List of Lemmas

List of Theorems

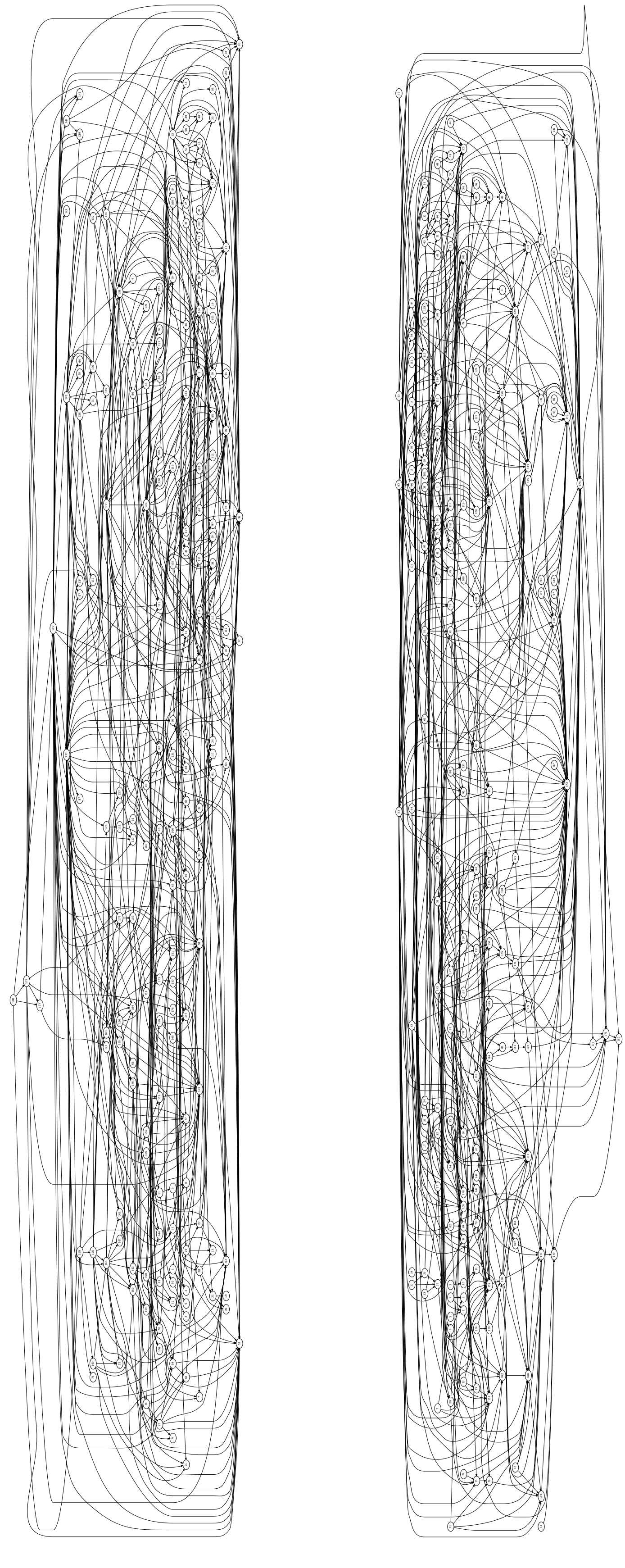

Figure 3: Dependency graph (both ways). All dependencies are detailed in Appendices B and C.

B Depends directly from...

Definition 2 (*supremum*) has no direct dependency.

Lemma 3 (*finite supremum*) has no direct dependency.

Lemma 4 (*discrete lower accumulation*) has no direct dependency.

Lemma 5 (*supremum is positive scalar multiplicative*) depends directly from:
Definition 2 (*supremum*).

Definition 7 (*maximum*) has no direct dependency.

Lemma 8 (*finite maximum*) depends directly from:
Definition 2 (*supremum*),
Lemma 3 (*finite supremum*),
Definition 7 (*maximum*).

Definition 9 (*infimum*) has no direct dependency.

Lemma 10 (*duality infimum-supremum*) depends directly from:
Definition 2 (*supremum*),
Definition 9 (*infimum*).

Lemma 11 (*finite infimum*) depends directly from:
Lemma 3 (*finite supremum*),
Lemma 10 (*duality infimum-supremum*).

Lemma 12 (*discrete upper accumulation*) depends directly from:
Lemma 4 (*discrete lower accumulation*).

Lemma 13 (*finite infimum discrete*) depends directly from:
Lemma 11 (*finite infimum*),
Lemma 12 (*discrete upper accumulation*).

Definition 14 (*minimum*) has no direct dependency.

Lemma 15 (*finite minimum*) depends directly from:
Lemma 8 (*finite maximum*),
Lemma 10 (*duality infimum-supremum*),
Definition 14 (*minimum*).

Definition 16 (*distance*) has no direct dependency.

Definition 17 (*metric space*) has no direct dependency.

Lemma 18 (*iterated triangle inequality*) depends directly from:
Definition 16 (*distance*).

Definition 19 (*closed ball*) has no direct dependency.

Definition 20 (*sphere*) has no direct dependency.

Definition 21 (*open subset*) has no direct dependency.

Definition 22 (*closed subset*) has no direct dependency.

Lemma 23 (*equivalent definition of closed subset*) depends directly from:
Definition 21 (*open subset*),
Definition 22 (*closed subset*).

Lemma 24 (*singleton is closed*) depends directly from:
Definition 16 (*distance*),
Lemma 23 (*equivalent definition of closed subset*).

Definition 25 (*closure*) has no direct dependency.

Definition 26 (*convergent sequence*) has no direct dependency.

Lemma 27 (*variant of point separation*) depends directly from:
Definition 16 (*distance*).

Lemma 28 (*limit is unique*) depends directly from:
Definition 16 (*distance*),
Definition 26 (*convergent sequence*),
Lemma 27 (*variant of point separation*).

Lemma 29 (*closure is limit of sequences*) depends directly from:
Definition 16 (*distance*),
Definition 19 (*closed ball*),
Definition 25 (*closure*),
Definition 26 (*convergent sequence*).

Lemma 30 (*closed equals closure*) depends directly from:
Definition 22 (*closed subset*),
Lemma 23 (*equivalent definition of closed subset*),
Definition 25 (*closure*).

Lemma 31 (*closed is limit of sequences*) depends directly from:
Definition 25 (*closure*),
Lemma 29 (*closure is limit of sequences*),
Lemma 30 (*closed equals closure*).

Definition 32 (*stationary sequence*) has no direct dependency.

Lemma 33 (*stationary sequence is convergent*) depends directly from:
Definition 16 (*distance*),
Definition 26 (*convergent sequence*),
Definition 32 (*stationary sequence*).

Definition 34 (*Cauchy sequence*) has no direct dependency.

Lemma 35 (*equivalent definition of Cauchy sequence*) depends directly from:
Definition 16 (*distance*),
Definition 34 (*Cauchy sequence*).

Lemma 36 (*convergent sequence is Cauchy*) depends directly from:
Definition 16 (*distance*),
Definition 26 (*convergent sequence*),
Lemma 28 (*limit is unique*),
Definition 34 (*Cauchy sequence*).

Definition 37 (*complete subset*) has no direct dependency.

Definition 38 (*complete metric space*) has no direct dependency.

Lemma 39 (*closed subset of complete is complete*) depends directly from:
Lemma 29 (*closure is limit of sequences*),
Lemma 30 (*closed equals closure*),
Definition 37 (*complete subset*),
Definition 38 (*complete metric space*).

Definition 42 (*continuity in a point*) has no direct dependency.

Definition 43 (*pointwise continuity*) has no direct dependency.

Lemma 44 (*compatibility of limit with continuous functions*) depends directly from:
Definition 26 (*convergent sequence*),
Definition 42 (*continuity in a point*).

Definition 45 (*uniform continuity*) has no direct dependency.

Definition 46 (*Lipschitz continuity*) has no direct dependency.

Theorem 47 (*equivalent definition of Lipschitz continuity*) depends directly from:
Definition 16 (*distance*),
Definition 46 (*Lipschitz continuity*).

Definition 48 (*contraction*) has no direct dependency.

Lemma 49 (*uniform continuous is continuous*) depends directly from:
Definition 42 (*continuity in a point*),
Definition 43 (*pointwise continuity*),
Definition 45 (*uniform continuity*).

Lemma 50 (*zero-Lipschitz continuous is constant*) depends directly from:
Definition 16 (*distance*),
Definition 46 (*Lipschitz continuity*).

Lemma 51 (*Lipschitz continuous is uniform continuous*) depends directly from:
Definition 45 (*uniform continuity*),
Definition 46 (*Lipschitz continuity*),
Lemma 50 (*zero-Lipschitz continuous is constant*).

Definition 52 (*iterated function sequence*) has no direct dependency.

Lemma 53 (*stationary iterated function sequence*) depends directly from:
Definition 32 (*stationary sequence*),
Definition 52 (*iterated function sequence*).

Lemma 54 (*iterate Lipschitz continuous mapping*) depends directly from:
Definition 46 (*Lipschitz continuity*),
Definition 52 (*iterated function sequence*).

Lemma 55 (*convergent iterated function sequence*) depends directly from:
Definition 26 (*convergent sequence*),
Lemma 28 (*limit is unique*),
Definition 32 (*stationary sequence*),
Lemma 33 (*stationary sequence is convergent*),
Definition 46 (*Lipschitz continuity*),
Lemma 50 (*zero-Lipschitz continuous is constant*),
Definition 52 (*iterated function sequence*).

Theorem 56 (*fixed point*) depends directly from:
Definition 16 (*distance*),
Lemma 18 (*iterated triangle inequality*),
Definition 32 (*stationary sequence*),
Lemma 33 (*stationary sequence is convergent*),
Lemma 35 (*equivalent definition of Cauchy sequence*),
Definition 37 (*complete subset*),
Definition 38 (*complete metric space*),

Definition 46 (*Lipschitz continuity*),
Definition 48 (*contraction*),
Lemma 50 (*zero-Lipschitz continuous is constant*),
Lemma 53 (*stationary iterated function sequence*),
Lemma 54 (*iterate Lipschitz continuous mapping*),
Lemma 55 (*convergent iterated function sequence*).

Definition 58 (*vector space*) has no direct dependency.

Definition 61 (*set of mappings to space*) has no direct dependency.

Definition 62 (*linear map*) has no direct dependency.

Definition 63 (*set of linear maps*) has no direct dependency.

Definition 64 (*linear form*) has no direct dependency.

Definition 65 (*bilinear map*) has no direct dependency.

Definition 66 (*bilinear form*) has no direct dependency.

Definition 67 (*set of bilinear forms*) has no direct dependency.

Lemma 68 (*zero times yields zero*) depends directly from:
Definition 58 (*vector space*).

Lemma 69 (*minus times yields opposite vector*) depends directly from:
Definition 58 (*vector space*),
Lemma 68 (*zero times yields zero*).

Definition 70 (*vector subtraction*) has no direct dependency.

Definition 71 (*scalar division*) has no direct dependency.

Lemma 72 (*times zero yields zero*) depends directly from:
Definition 58 (*vector space*),
Definition 70 (*vector subtraction*).

Lemma 73 (*zero-product property*) depends directly from:
Definition 58 (*vector space*),
Lemma 68 (*zero times yields zero*),
Lemma 72 (*times zero yields zero*).

Definition 74 (*subspace*) has no direct dependency.

Lemma 77 (*trivial subspaces*) depends directly from:
Definition 74 (*subspace*).

Lemma 78 (*closed under vector operations is subspace*) depends directly from:
Definition 58 (*vector space*),
Lemma 69 (*minus times yields opposite vector*),
Definition 74 (*subspace*).

Lemma 79 (*closed under linear combination is subspace*) depends directly from:
Definition 58 (*vector space*),
Lemma 73 (*zero-product property*),
Lemma 78 (*closed under vector operations is subspace*).

Definition 80 (*linear span*) has no direct dependency.

Definition 81 (*sum of subspaces*) has no direct dependency.

Definition 82 (*finite dimensional subspace*) has no direct dependency.

Definition 83 (*direct sum of subspaces*) has no direct dependency.

Lemma 84 (*equivalent definitions of direct sum*) depends directly from:
Definition 58 (*vector space*),
Lemma 69 (*minus times yields opposite vector*),
Definition 70 (*vector subtraction*),
Lemma 78 (*closed under vector operations is subspace*),
Definition 83 (*direct sum of subspaces*).

Lemma 85 (*direct sum with linear span*) depends directly from:
Lemma 73 (*zero-product property*),
Lemma 78 (*closed under vector operations is subspace*),
Definition 80 (*linear span*),
Lemma 84 (*equivalent definitions of direct sum*).

Definition 86 (*product vector operations*) has no direct dependency.

Lemma 87 (*product is space*) depends directly from:
Definition 58 (*vector space*),
Definition 86 (*product vector operations*).

Definition 88 (*inherited vector operations*) has no direct dependency.

Lemma 90 (*space of mappings to a space*) depends directly from:
Definition 58 (*vector space*),
Definition 61 (*set of mappings to space*),
Definition 88 (*inherited vector operations*).

Lemma 91 (*linear map preserves zero*) depends directly from:
Definition 62 (*linear map*),
Lemma 68 (*zero times yields zero*).

Lemma 92 (*linear map preserves linear combinations*) depends directly from:
Definition 58 (*vector space*),
Definition 62 (*linear map*),
Lemma 68 (*zero times yields zero*).

Lemma 93 (*space of linear maps*) depends directly from:
Definition 58 (*vector space*),
Definition 62 (*linear map*),
Definition 63 (*set of linear maps*),
Lemma 72 (*times zero yields zero*),
Lemma 79 (*closed under linear combination is subspace*),
Definition 88 (*inherited vector operations*),
Lemma 90 (*space of mappings to a space*).

Definition 94 (*identity map*) has no direct dependency.

Lemma 95 (*identity map is linear map*) depends directly from:
Definition 62 (*linear map*),
Definition 94 (*identity map*).

Lemma 96 (*composition of linear maps is bilinear*) depends directly from:
Definition 62 (*linear map*),
Definition 65 (*bilinear map*),
Lemma 87 (*product is space*),

Definition 88 (*inherited vector operations*),
Lemma 92 (*linear map preserves linear combinations*),
Lemma 93 (*space of linear maps*).

Definition 97 (*isomorphism*) has no direct dependency.

Definition 98 (*kernel*) has no direct dependency.

Lemma 99 (*kernel is subspace*) depends directly from:
Definition 58 (*vector space*),
Lemma 72 (*times zero yields zero*),
Lemma 79 (*closed under linear combination is subspace*),
Lemma 91 (*linear map preserves zero*),
Lemma 92 (*linear map preserves linear combinations*),
Definition 98 (*kernel*).

Lemma 100 (*injective linear map has zero kernel*) depends directly from:
Definition 58 (*vector space*),
Definition 62 (*linear map*),
Definition 70 (*vector subtraction*),
Lemma 91 (*linear map preserves zero*),
Definition 98 (*kernel*).

Lemma 101 (*K is space*) has no direct dependency.

Definition 102 (*norm*) has no direct dependency.

Definition 104 (*normed vector space*) has no direct dependency.

Lemma 105 (*K is normed vector space*) depends directly from:
Definition 102 (*norm*),
Definition 104 (*normed vector space*).

Lemma 106 (*norm preserves zero*) depends directly from:
Definition 58 (*vector space*),
Lemma 68 (*zero times yields zero*),
Definition 102 (*norm*),
Definition 104 (*normed vector space*).

Lemma 107 (*norm is nonnegative*) depends directly from:
Definition 58 (*vector space*),
Definition 102 (*norm*),
Definition 104 (*normed vector space*).

Lemma 108 (*normalization by nonzero*) depends directly from:
Definition 71 (*scalar division*),
Definition 102 (*norm*),
Lemma 107 (*norm is nonnegative*).

Definition 109 (*distance associated with norm*) has no direct dependency.

Lemma 111 (*norm gives distance*) depends directly from:
Definition 16 (*distance*),
Definition 17 (*metric space*),
Definition 70 (*vector subtraction*),
Definition 102 (*norm*),
Lemma 107 (*norm is nonnegative*),
Definition 109 (*distance associated with norm*).

Lemma 112 (*linear span is closed*) depends directly from:
Lemma 24 (*singleton is closed*),
Definition 26 (*convergent sequence*),
Lemma 28 (*limit is unique*),
Lemma 31 (*closed is limit of sequences*),
Definition 34 (*Cauchy sequence*),
Lemma 36 (*convergent sequence is Cauchy*),
Definition 37 (*complete subset*),
Definition 58 (*vector space*),
Definition 70 (*vector subtraction*),
Definition 80 (*linear span*),
Definition 102 (*norm*),
Lemma 107 (*norm is nonnegative*),
Definition 109 (*distance associated with norm*).

Definition 113 (*closed unit ball*) has no direct dependency.

Lemma 114 (*equivalent definition of closed unit ball*) depends directly from:
Definition 19 (*closed ball*),
Definition 109 (*distance associated with norm*),
Lemma 111 (*norm gives distance*),
Definition 113 (*closed unit ball*).

Definition 115 (*unit sphere*) has no direct dependency.

Lemma 116 (*equivalent definition of unit sphere*) depends directly from:
Definition 20 (*sphere*),
Definition 109 (*distance associated with norm*),
Lemma 111 (*norm gives distance*),
Definition 115 (*unit sphere*).

Lemma 117 (*zero on unit sphere is zero*) depends directly from:
Definition 58 (*vector space*),
Definition 62 (*linear map*),
Lemma 72 (*times zero yields zero*),
Lemma 91 (*linear map preserves zero*),
Lemma 108 (*normalization by nonzero*),
Lemma 116 (*equivalent definition of unit sphere*).

Lemma 118 (*reverse triangle inequality*) depends directly from:
Definition 102 (*norm*),
Definition 104 (*normed vector space*).

Lemma 119 (*norm is one-Lipschitz continuous*) depends directly from:
Definition 46 (*Lipschitz continuity*),
Definition 109 (*distance associated with norm*),
Lemma 118 (*reverse triangle inequality*).

Lemma 120 (*norm is uniformly continuous*) depends directly from:
Lemma 51 (*Lipschitz continuous is uniform continuous*),
Lemma 119 (*norm is one-Lipschitz continuous*).

Lemma 121 (*norm is continuous*) depends directly from:
Lemma 49 (*uniform continuous is continuous*),
Lemma 120 (*norm is uniformly continuous*).

Definition 122 (*linear isometry*) has no direct dependency.

Lemma 123 (*identity map is linear isometry*) depends directly from:
Definition 94 (*identity map*),
Lemma 95 (*identity map is linear map*),
Definition 122 (*linear isometry*).

Definition 124 (*product norm*) has no direct dependency.

Lemma 127 (*product is normed vector space*) depends directly from:
Definition 86 (*product vector operations*),
Lemma 87 (*product is space*),
Definition 102 (*norm*),
Definition 104 (*normed vector space*),
Lemma 107 (*norm is nonnegative*),
Definition 124 (*product norm*).

Lemma 128 (*vector addition is continuous*) depends directly from:
Definition 42 (*continuity in a point*),
Definition 43 (*pointwise continuity*),
Definition 58 (*vector space*),
Definition 86 (*product vector operations*),
Definition 102 (*norm*),
Definition 109 (*distance associated with norm*),
Definition 124 (*product norm*),
Lemma 127 (*product is normed vector space*).

Lemma 129 (*scalar multiplication is continuous*) depends directly from:
Definition 16 (*distance*),
Definition 42 (*continuity in a point*),
Definition 43 (*pointwise continuity*),
Definition 58 (*vector space*),
Lemma 68 (*zero times yields zero*),
Definition 102 (*norm*),
Definition 109 (*distance associated with norm*).

Lemma 133 (*norm of image of unit vector*) depends directly from:
Definition 62 (*linear map*),
Definition 102 (*norm*),
Lemma 107 (*norm is nonnegative*),
Lemma 108 (*normalization by nonzero*).

Lemma 134 (*norm of image of unit sphere*) depends directly from:
Definition 102 (*norm*),
Lemma 106 (*norm preserves zero*),
Lemma 116 (*equivalent definition of unit sphere*),
Lemma 133 (*norm of image of unit vector*).

Definition 135 (*operator norm*) has no direct dependency.

Lemma 137 (*equivalent definition of operator norm*) depends directly from:
Lemma 134 (*norm of image of unit sphere*),
Definition 135 (*operator norm*).

Lemma 138 (*operator norm is nonnegative*) depends directly from:
Definition 2 (*supremum*),
Lemma 107 (*norm is nonnegative*),
Lemma 137 (*equivalent definition of operator norm*).

Definition 139 (*bounded linear map*) has no direct dependency.

Definition 140 (*linear map bounded on unit ball*) has no direct dependency.

Definition 141 (*linear map bounded on unit sphere*) has no direct dependency.

Theorem 142 (*continuous linear map*) depends directly from:
Definition 2 (*supremum*),
Lemma 3 (*finite supremum*),
Definition 42 (*continuity in a point*),
Definition 43 (*pointwise continuity*),
Definition 46 (*Lipschitz continuity*),
Lemma 49 (*uniform continuous is continuous*),
Lemma 51 (*Lipschitz continuous is uniform continuous*),
Definition 62 (*linear map*),
Definition 70 (*vector subtraction*),
Lemma 91 (*linear map preserves zero*),
Definition 102 (*norm*),
Lemma 106 (*norm preserves zero*),
Lemma 114 (*equivalent definition of closed unit ball*),
Lemma 116 (*equivalent definition of unit sphere*),
Definition 135 (*operator norm*),
Lemma 137 (*equivalent definition of operator norm*),
Lemma 138 (*operator norm is nonnegative*),
Definition 139 (*bounded linear map*),
Definition 140 (*linear map bounded on unit ball*),
Definition 141 (*linear map bounded on unit sphere*).

Definition 143 (*set of continuous linear maps*) has no direct dependency.

Lemma 144 (*finite operator norm is continuous*) depends directly from:
Definition 2 (*supremum*),
Definition 139 (*bounded linear map*),
Definition 140 (*linear map bounded on unit ball*),
Definition 141 (*linear map bounded on unit sphere*),
Theorem 142 (*continuous linear map*),
Definition 143 (*set of continuous linear maps*).

Lemma 145 (*linear isometry is continuous*) depends directly from:
Definition 122 (*linear isometry*),
Lemma 144 (*finite operator norm is continuous*).

Lemma 146 (*identity map is continuous*) depends directly from:
Lemma 123 (*identity map is linear isometry*),
Lemma 145 (*linear isometry is continuous*).

Theorem 147 (*normed vector space of continuous linear maps*) depends directly from:
Definition 2 (*supremum*),
Lemma 5 (*supremum is positive scalar multiplicative*),
Lemma 78 (*closed under vector operations is subspace*),
Definition 88 (*inherited vector operations*),
Definition 102 (*norm*),
Definition 104 (*normed vector space*),
Lemma 107 (*norm is nonnegative*),
Lemma 117 (*zero on unit sphere is zero*),
Lemma 137 (*equivalent definition of operator norm*),
Definition 141 (*linear map bounded on unit sphere*),

Definition 143 (*set of continuous linear maps*),
Lemma 144 (*finite operator norm is continuous*).

Lemma 148 (*operator norm estimation*) depends directly from:
Definition 2 (*supremum*),
Lemma 91 (*linear map preserves zero*),
Definition 102 (*norm*),
Lemma 106 (*norm preserves zero*),
Lemma 107 (*norm is nonnegative*),
Definition 135 (*operator norm*),
Theorem 147 (*normed vector space of continuous linear maps*).

Lemma 149 (*continuous linear maps have closed kernel*) depends directly from:
Lemma 24 (*singleton is closed*),
Definition 98 (*kernel*).

Lemma 150 (*compatibility of composition with continuity*) depends directly from:
Lemma 96 (*composition of linear maps is bilinear*),
Lemma 116 (*equivalent definition of unit sphere*),
Lemma 144 (*finite operator norm is continuous*),
Lemma 148 (*operator norm estimation*).

Lemma 151 (*complete normed vector space of continuous linear maps*) depends directly from:
Definition 26 (*convergent sequence*),
Lemma 33 (*stationary sequence is convergent*),
Definition 34 (*Cauchy sequence*),
Definition 37 (*complete subset*),
Definition 38 (*complete metric space*),
Lemma 44 (*compatibility of limit with continuous functions*),
Definition 58 (*vector space*),
Definition 62 (*linear map*),
Definition 88 (*inherited vector operations*),
Lemma 91 (*linear map preserves zero*),
Lemma 92 (*linear map preserves linear combinations*),
Definition 102 (*norm*),
Definition 109 (*distance associated with norm*),
Lemma 111 (*norm gives distance*),
Lemma 121 (*norm is continuous*),
Lemma 128 (*vector addition is continuous*),
Lemma 129 (*scalar multiplication is continuous*),
Definition 135 (*operator norm*),
Definition 139 (*bounded linear map*),
Theorem 142 (*continuous linear map*),
Lemma 144 (*finite operator norm is continuous*),
Theorem 147 (*normed vector space of continuous linear maps*).

Definition 152 (*topological dual*) has no direct dependency.

Definition 153 (*dual norm*) has no direct dependency.

Lemma 154 (*topological dual is complete normed vector space*) depends directly from:
Lemma 105 (*K is normed vector space*),
Theorem 147 (*normed vector space of continuous linear maps*),
Lemma 151 (*complete normed vector space of continuous linear maps*),
Definition 153 (*dual norm*).

Definition 155 (*bra-ket notation*) has no direct dependency.

Lemma 156 (*bra-ket is bilinear map*) depends directly from:
Definition 62 (*linear map*),
Definition 65 (*bilinear map*),
Lemma 87 (*product is space*),
Definition 88 (*inherited vector operations*),
Lemma 101 (*K is space*),
Lemma 154 (*topological dual is complete normed vector space*),
Definition 155 (*bra-ket notation*).

Definition 157 (*bounded bilinear form*) has no direct dependency.

Lemma 158 (*representation for bounded bilinear form*) depends directly from:
Definition 58 (*vector space*),
Definition 64 (*linear form*),
Definition 65 (*bilinear map*),
Definition 66 (*bilinear form*),
Definition 67 (*set of bilinear forms*),
Definition 70 (*vector subtraction*),
Definition 88 (*inherited vector operations*),
Lemma 92 (*linear map preserves linear combinations*),
Lemma 101 (*K is space*),
Definition 104 (*normed vector space*),
Lemma 107 (*norm is nonnegative*),
Lemma 116 (*equivalent definition of unit sphere*),
Definition 139 (*bounded linear map*),
Lemma 144 (*finite operator norm is continuous*),
Theorem 147 (*normed vector space of continuous linear maps*),
Definition 152 (*topological dual*),
Definition 153 (*dual norm*),
Lemma 154 (*topological dual is complete normed vector space*),
Definition 155 (*bra-ket notation*),
Lemma 156 (*bra-ket is bilinear map*),
Definition 157 (*bounded bilinear form*).

Definition 159 (*coercive bilinear form*) has no direct dependency.

Lemma 160 (*coercivity constant is less than continuity constant*) depends directly from:
Definition 102 (*norm*),
Lemma 107 (*norm is nonnegative*),
Definition 157 (*bounded bilinear form*),
Definition 159 (*coercive bilinear form*).

Definition 161 (*inner product*) has no direct dependency.

Definition 165 (*inner product space*) has no direct dependency.

Lemma 166 (*inner product subspace*) depends directly from:
Definition 74 (*subspace*),
Definition 161 (*inner product*),
Definition 165 (*inner product space*).

Lemma 167 (*inner product with zero is zero*) depends directly from:
Definition 58 (*vector space*),
Definition 65 (*bilinear map*),
Definition 70 (*vector subtraction*),
Definition 161 (*inner product*).

Lemma 168 (*square expansion plus*) depends directly from:
Definition 65 (*bilinear map*),
Definition 161 (*inner product*).

Lemma 169 (*square expansion minus*) depends directly from:
Definition 65 (*bilinear map*),
Definition 70 (*vector subtraction*),
Definition 161 (*inner product*),
Lemma 168 (*square expansion plus*).

Lemma 170 (*parallelogram identity*) depends directly from:
Lemma 168 (*square expansion plus*),
Lemma 169 (*square expansion minus*).

Lemma 171 (*Cauchy–Schwarz inequality*) depends directly from:
Definition 65 (*bilinear map*),
Definition 161 (*inner product*),
Lemma 168 (*square expansion plus*).

Definition 172 (*square root of inner square*) has no direct dependency.

Lemma 174 (*squared norm*) depends directly from:
Definition 161 (*inner product*),
Definition 172 (*square root of inner square*).

Lemma 175 (*Cauchy–Schwarz inequality with norms*) depends directly from:
Definition 161 (*inner product*),
Lemma 171 (*Cauchy–Schwarz inequality*),
Definition 172 (*square root of inner square*).

Lemma 176 (*triangle inequality*) depends directly from:
Definition 161 (*inner product*),
Lemma 168 (*square expansion plus*),
Lemma 174 (*squared norm*),
Lemma 175 (*Cauchy–Schwarz inequality with norms*).

Lemma 177 (*inner product gives norm*) depends directly from:
Definition 102 (*norm*),
Definition 104 (*normed vector space*),
Definition 161 (*inner product*),
Definition 172 (*square root of inner square*),
Lemma 176 (*triangle inequality*).

Definition 179 (*convex subset*) has no direct dependency.

Theorem 180 (*orthogonal projection onto nonempty complete convex*) depends directly from:
Definition 9 (*infimum*),
Lemma 13 (*finite infimum discrete*),
Definition 14 (*minimum*),
Definition 34 (*Cauchy sequence*),
Definition 37 (*complete subset*),
Lemma 44 (*compatibility of limit with continuous functions*),
Definition 58 (*vector space*),
Definition 70 (*vector subtraction*),
Definition 71 (*scalar division*),
Definition 102 (*norm*),

Lemma 107 (*norm is nonnegative*),
Lemma 121 (*norm is continuous*),
Definition 165 (*inner product space*),
Lemma 170 (*parallelogram identity*),
Lemma 174 (*squared norm*),
Definition 179 (*convex subset*).

Lemma 181 (*characterization of orthogonal projection onto convex*) depends directly from:
Definition 9 (*infimum*),
Definition 14 (*minimum*),
Lemma 15 (*finite minimum*),
Definition 58 (*vector space*),
Definition 65 (*bilinear map*),
Definition 70 (*vector subtraction*),
Lemma 107 (*norm is nonnegative*),
Definition 161 (*inner product*),
Definition 165 (*inner product space*),
Lemma 168 (*square expansion plus*),
Lemma 174 (*squared norm*),
Definition 179 (*convex subset*).

Lemma 182 (*subspace is convex*) depends directly from:
Lemma 79 (*closed under linear combination is subspace*),
Definition 179 (*convex subset*).

Theorem 183 (*orthogonal projection onto complete subspace*) depends directly from:
Definition 58 (*vector space*),
Definition 74 (*subspace*),
Theorem 180 (*orthogonal projection onto nonempty complete convex*),
Lemma 182 (*subspace is convex*).

Definition 184 (*orthogonal projection onto complete subspace*) depends directly from:
Theorem 183 (*orthogonal projection onto complete subspace*).

Lemma 185 (*characterization of orthogonal projection onto subspace*) depends directly from:
Definition 58 (*vector space*),
Definition 65 (*bilinear map*),
Definition 70 (*vector subtraction*),
Definition 74 (*subspace*),
Definition 161 (*inner product*),
Lemma 181 (*characterization of orthogonal projection onto convex*),
Lemma 182 (*subspace is convex*).

Lemma 186 (*orthogonal projection is continuous linear map*) depends directly from:
Definition 46 (*Lipschitz continuity*),
Definition 58 (*vector space*),
Definition 65 (*bilinear map*),
Definition 74 (*subspace*),
Lemma 92 (*linear map preserves linear combinations*),
Definition 102 (*norm*),
Lemma 107 (*norm is nonnegative*),
Definition 161 (*inner product*),
Lemma 174 (*squared norm*),
Lemma 175 (*Cauchy–Schwarz inequality with norms*),
Theorem 183 (*orthogonal projection onto complete subspace*),

Definition 184 (*orthogonal projection onto complete subspace*),
Lemma 185 (*characterization of orthogonal projection onto subspace*).

Definition 187 (*orthogonal complement*) has no direct dependency.

Lemma 188 (*trivial orthogonal complements*) depends directly from:
Lemma 77 (*trivial subspaces*),
Definition 161 (*inner product*),
Lemma 167 (*inner product with zero is zero*),
Definition 187 (*orthogonal complement*).

Lemma 189 (*orthogonal complement is subspace*) depends directly from:
Definition 65 (*bilinear map*),
Lemma 79 (*closed under linear combination is subspace*),
Definition 161 (*inner product*),
Lemma 167 (*inner product with zero is zero*),
Definition 187 (*orthogonal complement*).

Lemma 190 (*zero intersection with orthogonal complement*) depends directly from:
Definition 161 (*inner product*),
Definition 187 (*orthogonal complement*).

Theorem 191 (*direct sum with orthogonal complement when complete*) depends directly from:
Definition 58 (*vector space*),
Definition 65 (*bilinear map*),
Definition 81 (*sum of subspaces*),
Definition 83 (*direct sum of subspaces*),
Lemma 84 (*equivalent definitions of direct sum*),
Definition 161 (*inner product*),
Lemma 167 (*inner product with zero is zero*),
Theorem 183 (*orthogonal projection onto complete subspace*),
Definition 184 (*orthogonal projection onto complete subspace*),
Lemma 185 (*characterization of orthogonal projection onto subspace*),
Definition 187 (*orthogonal complement*),
Lemma 190 (*zero intersection with orthogonal complement*).

Lemma 192 (*sum is orthogonal sum*) depends directly from:
Definition 58 (*vector space*),
Lemma 79 (*closed under linear combination is subspace*),
Definition 80 (*linear span*),
Definition 81 (*sum of subspaces*),
Theorem 183 (*orthogonal projection onto complete subspace*),
Theorem 191 (*direct sum with orthogonal complement when complete*).

Lemma 193 (*sum of complete subspace and linear span is closed*) depends directly from:
Lemma 31 (*closed is limit of sequences*),
Lemma 44 (*compatibility of limit with continuous functions*),
Lemma 79 (*closed under linear combination is subspace*),
Definition 80 (*linear span*),
Definition 81 (*sum of subspaces*),
Lemma 85 (*direct sum with linear span*),
Lemma 112 (*linear span is closed*),
Lemma 146 (*identity map is continuous*),
Theorem 147 (*normed vector space of continuous linear maps*),
Theorem 183 (*orthogonal projection onto complete subspace*),

Lemma 186 (*orthogonal projection is continuous linear map*),
Lemma 190 (*zero intersection with orthogonal complement*),
Theorem 191 (*direct sum with orthogonal complement when complete*).

Definition 194 (*Hilbert space*) depends directly from:
Lemma 111 (*norm gives distance*),
Definition 172 (*square root of inner square*),
Lemma 177 (*inner product gives norm*).

Lemma 195 (*closed Hilbert subspace*) depends directly from:
Lemma 39 (*closed subset of complete is complete*),
Definition 74 (*subspace*),
Lemma 166 (*inner product subspace*),
Definition 194 (*Hilbert space*).

Theorem 196 (*Riesz–Fréchet*) depends directly from:
Definition 2 (*supremum*),
Definition 58 (*vector space*),
Definition 62 (*linear map*),
Definition 64 (*linear form*),
Definition 65 (*bilinear map*),
Definition 70 (*vector subtraction*),
Definition 71 (*scalar division*),
Lemma 73 (*zero-product property*),
Lemma 78 (*closed under vector operations is subspace*),
Lemma 91 (*linear map preserves zero*),
Lemma 92 (*linear map preserves linear combinations*),
Definition 97 (*isomorphism*),
Definition 98 (*kernel*),
Lemma 99 (*kernel is subspace*),
Lemma 100 (*injective linear map has zero kernel*),
Definition 102 (*norm*),
Definition 104 (*normed vector space*),
Lemma 106 (*norm preserves zero*),
Lemma 107 (*norm is nonnegative*),
Lemma 108 (*normalization by nonzero*),
Definition 122 (*linear isometry*),
Definition 135 (*operator norm*),
Definition 139 (*bounded linear map*),
Theorem 142 (*continuous linear map*),
Lemma 144 (*finite operator norm is continuous*),
Lemma 145 (*linear isometry is continuous*),
Lemma 149 (*continuous linear maps have closed kernel*),
Definition 152 (*topological dual*),
Definition 153 (*dual norm*),
Lemma 154 (*topological dual is complete normed vector space*),
Definition 155 (*bra-ket notation*),
Lemma 156 (*bra-ket is bilinear map*),
Definition 161 (*inner product*),
Definition 165 (*inner product space*),
Lemma 167 (*inner product with zero is zero*),
Lemma 174 (*squared norm*),
Lemma 175 (*Cauchy–Schwarz inequality with norms*),
Theorem 183 (*orthogonal projection onto complete subspace*),
Definition 184 (*orthogonal projection onto complete subspace*),

Definition 187 (*orthogonal complement*),
Lemma 188 (*trivial orthogonal complements*),
Lemma 189 (*orthogonal complement is subspace*),
Theorem 191 (*direct sum with orthogonal complement when complete*),
Definition 194 (*Hilbert space*),
Lemma 195 (*closed Hilbert subspace*).

Lemma 197 (*compatible ρ for Lax–Milgram*) has no direct dependency.

Theorem 198 (*Lax–Milgram*) depends directly from:
Definition 46 (*Lipschitz continuity*),
Definition 48 (*contraction*),
Theorem 56 (*fixed point*),
Definition 58 (*vector space*),
Definition 65 (*bilinear map*),
Lemma 69 (*minus times yields opposite vector*),
Definition 70 (*vector subtraction*),
Lemma 73 (*zero-product property*),
Definition 102 (*norm*),
Definition 104 (*normed vector space*),
Lemma 106 (*norm preserves zero*),
Lemma 107 (*norm is nonnegative*),
Definition 122 (*linear isometry*),
Lemma 146 (*identity map is continuous*),
Theorem 147 (*normed vector space of continuous linear maps*),
Lemma 148 (*operator norm estimation*),
Lemma 150 (*compatibility of composition with continuity*),
Lemma 154 (*topological dual is complete normed vector space*),
Definition 157 (*bounded bilinear form*),
Lemma 158 (*representation for bounded bilinear form*),
Definition 159 (*coercive bilinear form*),
Lemma 160 (*coercivity constant is less than continuity constant*),
Definition 161 (*inner product*),
Lemma 168 (*square expansion plus*),
Lemma 177 (*inner product gives norm*),
Definition 187 (*orthogonal complement*),
Lemma 188 (*trivial orthogonal complements*),
Definition 194 (*Hilbert space*),
Theorem 196 (*Riesz–Fréchet*),
Lemma 197 (*compatible ρ for Lax–Milgram*).

Lemma 199 (*Galerkin orthogonality*) depends directly from:
Definition 65 (*bilinear map*),
Definition 74 (*subspace*).

Theorem 200 (*Lax–Milgram, closed subspace*) depends directly from:
Lemma 195 (*closed Hilbert subspace*),
Theorem 198 (*Lax–Milgram*).

Lemma 201 (*Céa*) depends directly from:
Definition 58 (*vector space*),
Definition 65 (*bilinear map*),
Definition 70 (*vector subtraction*),
Definition 102 (*norm*),
Lemma 106 (*norm preserves zero*),
Lemma 107 (*norm is nonnegative*),

Definition 157 (*bounded bilinear form*),
Definition 159 (*coercive bilinear form*),
Definition 161 (*inner product*),
Lemma 199 (*Galerkin orthogonality*),
Theorem 200 (*Lax–Milgram, closed subspace*).

Lemma 202 (*finite dimensional subspace in Hilbert space is closed*) depends directly from:
Lemma 39 (*closed subset of complete is complete*),
Definition 82 (*finite dimensional subspace*),
Lemma 112 (*linear span is closed*),
Lemma 192 (*sum is orthogonal sum*),
Lemma 193 (*sum of complete subspace and linear span is closed*),
Definition 194 (*Hilbert space*).

Theorem 203 (*Lax–Milgram–Céa, finite dimensional subspace*) depends directly from:
Theorem 198 (*Lax–Milgram*),
Theorem 200 (*Lax–Milgram, closed subspace*),
Lemma 201 (*Céa*),
Lemma 202 (*finite dimensional subspace in Hilbert space is closed*).

C Is a direct dependency of...

Definition 2 (*supremum*) is a direct dependency of:
Lemma 5 (*supremum is positive scalar multiplicative*),
Lemma 8 (*finite maximum*),
Lemma 10 (*duality infimum-supremum*),
Lemma 138 (*operator norm is nonnegative*),
Theorem 142 (*continuous linear map*),
Lemma 144 (*finite operator norm is continuous*),
Theorem 147 (*normed vector space of continuous linear maps*),
Lemma 148 (*operator norm estimation*),
Theorem 196 (*Riesz–Fréchet*).

Lemma 3 (*finite supremum*) is a direct dependency of:
Lemma 8 (*finite maximum*),
Lemma 11 (*finite infimum*),
Theorem 142 (*continuous linear map*).

Lemma 4 (*discrete lower accumulation*) is a direct dependency of:
Lemma 12 (*discrete upper accumulation*).

Lemma 5 (*supremum is positive scalar multiplicative*) is a direct dependency of:
Theorem 147 (*normed vector space of continuous linear maps*).

Definition 7 (*maximum*) is a direct dependency of:
Lemma 8 (*finite maximum*).

Lemma 8 (*finite maximum*) is a direct dependency of:
Lemma 15 (*finite minimum*).

Definition 9 (*infimum*) is a direct dependency of:
Lemma 10 (*duality infimum-supremum*),
Theorem 180 (*orthogonal projection onto nonempty complete convex*),
Lemma 181 (*characterization of orthogonal projection onto convex*).

Lemma 10 (*duality infimum-supremum*) is a direct dependency of:
Lemma 11 (*finite infimum*),
Lemma 15 (*finite minimum*).

Lemma 11 (*finite infimum*) is a direct dependency of:
Lemma 13 (*finite infimum discrete*).

Lemma 12 (*discrete upper accumulation*) is a direct dependency of:
Lemma 13 (*finite infimum discrete*).

Lemma 13 (*finite infimum discrete*) is a direct dependency of:
Theorem 180 (*orthogonal projection onto nonempty complete convex*).

Definition 14 (*minimum*) is a direct dependency of:
Lemma 15 (*finite minimum*),
Theorem 180 (*orthogonal projection onto nonempty complete convex*),
Lemma 181 (*characterization of orthogonal projection onto convex*).

Lemma 15 (*finite minimum*) is a direct dependency of:
Lemma 181 (*characterization of orthogonal projection onto convex*).

Definition 16 (*distance*) is a direct dependency of:
Lemma 18 (*iterated triangle inequality*),
Lemma 24 (*singleton is closed*),
Lemma 27 (*variant of point separation*),
Lemma 28 (*limit is unique*),
Lemma 29 (*closure is limit of sequences*),
Lemma 33 (*stationary sequence is convergent*),
Lemma 35 (*equivalent definition of Cauchy sequence*),
Lemma 36 (*convergent sequence is Cauchy*),
Theorem 47 (*equivalent definition of Lipschitz continuity*),
Lemma 50 (*zero-Lipschitz continuous is constant*),
Theorem 56 (*fixed point*),
Lemma 111 (*norm gives distance*),
Lemma 129 (*scalar multiplication is continuous*).

Definition 17 (*metric space*) is a direct dependency of:
Lemma 111 (*norm gives distance*).

Lemma 18 (*iterated triangle inequality*) is a direct dependency of:
Theorem 56 (*fixed point*).

Definition 19 (*closed ball*) is a direct dependency of:
Lemma 29 (*closure is limit of sequences*),
Lemma 114 (*equivalent definition of closed unit ball*).

Definition 20 (*sphere*) is a direct dependency of:
Lemma 116 (*equivalent definition of unit sphere*).

Definition 21 (*open subset*) is a direct dependency of:
Lemma 23 (*equivalent definition of closed subset*).

Definition 22 (*closed subset*) is a direct dependency of:
Lemma 23 (*equivalent definition of closed subset*),
Lemma 30 (*closed equals closure*).

Lemma 23 (*equivalent definition of closed subset*) is a direct dependency of:
Lemma 24 (*singleton is closed*),
Lemma 30 (*closed equals closure*).

Lemma 24 (*singleton is closed*) is a direct dependency of:
Lemma 112 (*linear span is closed*),
Lemma 149 (*continuous linear maps have closed kernel*).

Definition 25 (*closure*) is a direct dependency of:
Lemma 29 (*closure is limit of sequences*),
Lemma 30 (*closed equals closure*),
Lemma 31 (*closed is limit of sequences*).

Definition 26 (*convergent sequence*) is a direct dependency of:
Lemma 28 (*limit is unique*),
Lemma 29 (*closure is limit of sequences*),
Lemma 33 (*stationary sequence is convergent*),
Lemma 36 (*convergent sequence is Cauchy*),
Lemma 44 (*compatibility of limit with continuous functions*),
Lemma 55 (*convergent iterated function sequence*),
Lemma 112 (*linear span is closed*),
Lemma 151 (*complete normed vector space of continuous linear maps*).

Lemma 27 (*variant of point separation*) is a direct dependency of:
Lemma 28 (*limit is unique*).

Lemma 28 (*limit is unique*) is a direct dependency of:
Lemma 36 (*convergent sequence is Cauchy*),
Lemma 55 (*convergent iterated function sequence*),
Lemma 112 (*linear span is closed*).

Lemma 29 (*closure is limit of sequences*) is a direct dependency of:
Lemma 31 (*closed is limit of sequences*),
Lemma 39 (*closed subset of complete is complete*).

Lemma 30 (*closed equals closure*) is a direct dependency of:
Lemma 31 (*closed is limit of sequences*),
Lemma 39 (*closed subset of complete is complete*).

Lemma 31 (*closed is limit of sequences*) is a direct dependency of:
Lemma 112 (*linear span is closed*),
Lemma 193 (*sum of complete subspace and linear span is closed*).

Definition 32 (*stationary sequence*) is a direct dependency of:
Lemma 33 (*stationary sequence is convergent*),
Lemma 53 (*stationary iterated function sequence*),
Lemma 55 (*convergent iterated function sequence*),
Theorem 56 (*fixed point*).

Lemma 33 (*stationary sequence is convergent*) is a direct dependency of:
Lemma 55 (*convergent iterated function sequence*),
Theorem 56 (*fixed point*),
Lemma 151 (*complete normed vector space of continuous linear maps*).

Definition 34 (*Cauchy sequence*) is a direct dependency of:
Lemma 35 (*equivalent definition of Cauchy sequence*),
Lemma 36 (*convergent sequence is Cauchy*),
Lemma 112 (*linear span is closed*),
Lemma 151 (*complete normed vector space of continuous linear maps*),
Theorem 180 (*orthogonal projection onto nonempty complete convex*).

Lemma 35 (*equivalent definition of Cauchy sequence*) is a direct dependency of:
Theorem 56 (*fixed point*).

Lemma 36 (*convergent sequence is Cauchy*) is a direct dependency of:
Lemma 112 (*linear span is closed*).

Definition 37 (*complete subset*) is a direct dependency of:
Lemma 39 (*closed subset of complete is complete*),
Theorem 56 (*fixed point*),
Lemma 112 (*linear span is closed*),
Lemma 151 (*complete normed vector space of continuous linear maps*),
Theorem 180 (*orthogonal projection onto nonempty complete convex*).

Definition 38 (*complete metric space*) is a direct dependency of:
Lemma 39 (*closed subset of complete is complete*),
Theorem 56 (*fixed point*),
Lemma 151 (*complete normed vector space of continuous linear maps*).

Lemma 39 (*closed subset of complete is complete*) is a direct dependency of:
Lemma 195 (*closed Hilbert subspace*),
Lemma 202 (*finite dimensional subspace in Hilbert space is closed*).

Definition 42 (*continuity in a point*) is a direct dependency of:
Lemma 44 (*compatibility of limit with continuous functions*),
Lemma 49 (*uniform continuous is continuous*),
Lemma 128 (*vector addition is continuous*),
Lemma 129 (*scalar multiplication is continuous*),
Theorem 142 (*continuous linear map*).

Definition 43 (*pointwise continuity*) is a direct dependency of:
Lemma 49 (*uniform continuous is continuous*),
Lemma 128 (*vector addition is continuous*),
Lemma 129 (*scalar multiplication is continuous*),
Theorem 142 (*continuous linear map*).

Lemma 44 (*compatibility of limit with continuous functions*) is a direct dependency of:
Lemma 151 (*complete normed vector space of continuous linear maps*),
Theorem 180 (*orthogonal projection onto nonempty complete convex*),
Lemma 193 (*sum of complete subspace and linear span is closed*).

Definition 45 (*uniform continuity*) is a direct dependency of:
Lemma 49 (*uniform continuous is continuous*),
Lemma 51 (*Lipschitz continuous is uniform continuous*).

Definition 46 (*Lipschitz continuity*) is a direct dependency of:
Theorem 47 (*equivalent definition of Lipschitz continuity*),
Lemma 50 (*zero-Lipschitz continuous is constant*),
Lemma 51 (*Lipschitz continuous is uniform continuous*),
Lemma 54 (*iterate Lipschitz continuous mapping*),
Lemma 55 (*convergent iterated function sequence*),
Theorem 56 (*fixed point*),
Lemma 119 (*norm is one-Lipschitz continuous*),
Theorem 142 (*continuous linear map*),
Lemma 186 (*orthogonal projection is continuous linear map*),
Theorem 198 (*Lax–Milgram*).

Theorem 47 (*equivalent definition of Lipschitz continuity*)

Definition 48 (*contraction*) is a direct dependency of:
Theorem 56 (*fixed point*),
Theorem 198 (*Lax–Milgram*).

Lemma 49 (*uniform continuous is continuous*) is a direct dependency of:
Lemma 121 (*norm is continuous*),
Theorem 142 (*continuous linear map*).

Lemma 50 (*zero-Lipschitz continuous is constant*) is a direct dependency of:
Lemma 51 (*Lipschitz continuous is uniform continuous*),
Lemma 55 (*convergent iterated function sequence*),
Theorem 56 (*fixed point*).

Lemma 51 (*Lipschitz continuous is uniform continuous*) is a direct dependency of:
Lemma 120 (*norm is uniformly continuous*),
Theorem 142 (*continuous linear map*).

Definition 52 (*iterated function sequence*) is a direct dependency of:
Lemma 53 (*stationary iterated function sequence*),
Lemma 54 (*iterate Lipschitz continuous mapping*),
Lemma 55 (*convergent iterated function sequence*).

Lemma 53 (*stationary iterated function sequence*) is a direct dependency of:
Theorem 56 (*fixed point*).

Lemma 54 (*iterate Lipschitz continuous mapping*) is a direct dependency of:
Theorem 56 (*fixed point*).

Lemma 55 (*convergent iterated function sequence*) is a direct dependency of:
Theorem 56 (*fixed point*).

Theorem 56 (*fixed point*) is a direct dependency of:
Theorem 198 (*Lax–Milgram*).

Definition 58 (*vector space*) is a direct dependency of:
Lemma 68 (*zero times yields zero*),
Lemma 69 (*minus times yields opposite vector*),
Lemma 72 (*times zero yields zero*),
Lemma 73 (*zero-product property*),
Lemma 78 (*closed under vector operations is subspace*),
Lemma 79 (*closed under linear combination is subspace*),
Lemma 84 (*equivalent definitions of direct sum*),
Lemma 87 (*product is space*),
Lemma 90 (*space of mappings to a space*),
Lemma 92 (*linear map preserves linear combinations*),
Lemma 93 (*space of linear maps*),
Lemma 99 (*kernel is subspace*),
Lemma 100 (*injective linear map has zero kernel*),
Lemma 106 (*norm preserves zero*),
Lemma 107 (*norm is nonnegative*),
Lemma 112 (*linear span is closed*),
Lemma 117 (*zero on unit sphere is zero*),
Lemma 128 (*vector addition is continuous*),
Lemma 129 (*scalar multiplication is continuous*),
Lemma 151 (*complete normed vector space of continuous linear maps*),
Lemma 158 (*representation for bounded bilinear form*),
Lemma 167 (*inner product with zero is zero*),
Theorem 180 (*orthogonal projection onto nonempty complete convex*),
Lemma 181 (*characterization of orthogonal projection onto convex*),
Theorem 183 (*orthogonal projection onto complete subspace*),
Lemma 185 (*characterization of orthogonal projection onto subspace*),
Lemma 186 (*orthogonal projection is continuous linear map*),
Theorem 191 (*direct sum with orthogonal complement when complete*),
Lemma 192 (*sum is orthogonal sum*),
Theorem 196 (*Riesz–Fréchet*),
Theorem 198 (*Lax–Milgram*),
Lemma 201 (*Céa*).

Definition 61 (*set of mappings to space*) is a direct dependency of:
Lemma 90 (*space of mappings to a space*).

Definition 62 (*linear map*) is a direct dependency of:
Lemma 91 (*linear map preserves zero*),
Lemma 92 (*linear map preserves linear combinations*),
Lemma 93 (*space of linear maps*),
Lemma 95 (*identity map is linear map*),
Lemma 96 (*composition of linear maps is bilinear*),
Lemma 100 (*injective linear map has zero kernel*),

Lemma 117 (*zero on unit sphere is zero*),
Lemma 133 (*norm of image of unit vector*),
Theorem 142 (*continuous linear map*),
Lemma 151 (*complete normed vector space of continuous linear maps*),
Lemma 156 (*bra-ket is bilinear map*),
Theorem 196 (*Riesz–Fréchet*).

Definition 63 (*set of linear maps*) is a direct dependency of:
Lemma 93 (*space of linear maps*).

Definition 64 (*linear form*) is a direct dependency of:
Lemma 158 (*representation for bounded bilinear form*),
Theorem 196 (*Riesz–Fréchet*).

Definition 65 (*bilinear map*) is a direct dependency of:
Lemma 96 (*composition of linear maps is bilinear*),
Lemma 156 (*bra-ket is bilinear map*),
Lemma 158 (*representation for bounded bilinear form*),
Lemma 167 (*inner product with zero is zero*),
Lemma 168 (*square expansion plus*),
Lemma 169 (*square expansion minus*),
Lemma 171 (*Cauchy–Schwarz inequality*),
Lemma 181 (*characterization of orthogonal projection onto convex*),
Lemma 185 (*characterization of orthogonal projection onto subspace*),
Lemma 186 (*orthogonal projection is continuous linear map*),
Lemma 189 (*orthogonal complement is subspace*),
Theorem 191 (*direct sum with orthogonal complement when complete*),
Theorem 196 (*Riesz–Fréchet*),
Theorem 198 (*Lax–Milgram*),
Lemma 199 (*Galerkin orthogonality*),
Lemma 201 (*Céa*).

Definition 66 (*bilinear form*) is a direct dependency of:
Lemma 158 (*representation for bounded bilinear form*).

Definition 67 (*set of bilinear forms*) is a direct dependency of:
Lemma 158 (*representation for bounded bilinear form*).

Lemma 68 (*zero times yields zero*) is a direct dependency of:
Lemma 69 (*minus times yields opposite vector*),
Lemma 73 (*zero-product property*),
Lemma 91 (*linear map preserves zero*),
Lemma 92 (*linear map preserves linear combinations*),
Lemma 106 (*norm preserves zero*),
Lemma 129 (*scalar multiplication is continuous*).

Lemma 69 (*minus times yields opposite vector*) is a direct dependency of:
Lemma 78 (*closed under vector operations is subspace*),
Lemma 84 (*equivalent definitions of direct sum*),
Theorem 198 (*Lax–Milgram*).

Definition 70 (*vector subtraction*) is a direct dependency of:
Lemma 72 (*times zero yields zero*),
Lemma 84 (*equivalent definitions of direct sum*),
Lemma 100 (*injective linear map has zero kernel*),
Lemma 111 (*norm gives distance*),
Lemma 112 (*linear span is closed*),

Theorem 142 (*continuous linear map*),
Lemma 158 (*representation for bounded bilinear form*),
Lemma 167 (*inner product with zero is zero*),
Lemma 169 (*square expansion minus*),
Theorem 180 (*orthogonal projection onto nonempty complete convex*),
Lemma 181 (*characterization of orthogonal projection onto convex*),
Lemma 185 (*characterization of orthogonal projection onto subspace*),
Theorem 196 (*Riesz–Fréchet*),
Theorem 198 (*Lax–Milgram*),
Lemma 201 (*Céa*).

Definition 71 (*scalar division*) is a direct dependency of:
Lemma 108 (*normalization by nonzero*),
Theorem 180 (*orthogonal projection onto nonempty complete convex*),
Theorem 196 (*Riesz–Fréchet*).

Lemma 72 (*times zero yields zero*) is a direct dependency of:
Lemma 73 (*zero-product property*),
Lemma 93 (*space of linear maps*),
Lemma 99 (*kernel is subspace*),
Lemma 117 (*zero on unit sphere is zero*).

Lemma 73 (*zero-product property*) is a direct dependency of:
Lemma 79 (*closed under linear combination is subspace*),
Lemma 85 (*direct sum with linear span*),
Theorem 196 (*Riesz–Fréchet*),
Theorem 198 (*Lax–Milgram*).

Definition 74 (*subspace*) is a direct dependency of:
Lemma 77 (*trivial subspaces*),
Lemma 78 (*closed under vector operations is subspace*),
Lemma 166 (*inner product subspace*),
Theorem 183 (*orthogonal projection onto complete subspace*),
Lemma 185 (*characterization of orthogonal projection onto subspace*),
Lemma 186 (*orthogonal projection is continuous linear map*),
Lemma 195 (*closed Hilbert subspace*),
Lemma 199 (*Galerkin orthogonality*).

Lemma 77 (*trivial subspaces*) is a direct dependency of:
Lemma 188 (*trivial orthogonal complements*).

Lemma 78 (*closed under vector operations is subspace*) is a direct dependency of:
Lemma 79 (*closed under linear combination is subspace*),
Lemma 84 (*equivalent definitions of direct sum*),
Lemma 85 (*direct sum with linear span*),
Theorem 147 (*normed vector space of continuous linear maps*),
Theorem 196 (*Riesz–Fréchet*).

Lemma 79 (*closed under linear combination is subspace*) is a direct dependency of:
Lemma 93 (*space of linear maps*),
Lemma 99 (*kernel is subspace*),
Lemma 182 (*subspace is convex*),
Lemma 189 (*orthogonal complement is subspace*),
Lemma 192 (*sum is orthogonal sum*),
Lemma 193 (*sum of complete subspace and linear span is closed*).

Definition 80 (*linear span*) is a direct dependency of:
Lemma 85 (*direct sum with linear span*),
Lemma 112 (*linear span is closed*),
Lemma 192 (*sum is orthogonal sum*),
Lemma 193 (*sum of complete subspace and linear span is closed*).

Definition 81 (*sum of subspaces*) is a direct dependency of:
Theorem 191 (*direct sum with orthogonal complement when complete*),
Lemma 192 (*sum is orthogonal sum*),
Lemma 193 (*sum of complete subspace and linear span is closed*).

Definition 82 (*finite dimensional subspace*) is a direct dependency of:
Lemma 202 (*finite dimensional subspace in Hilbert space is closed*).

Definition 83 (*direct sum of subspaces*) is a direct dependency of:
Lemma 84 (*equivalent definitions of direct sum*),
Theorem 191 (*direct sum with orthogonal complement when complete*).

Lemma 84 (*equivalent definitions of direct sum*) is a direct dependency of:
Lemma 85 (*direct sum with linear span*),
Theorem 191 (*direct sum with orthogonal complement when complete*).

Lemma 85 (*direct sum with linear span*) is a direct dependency of:
Lemma 193 (*sum of complete subspace and linear span is closed*).

Definition 86 (*product vector operations*) is a direct dependency of:
Lemma 87 (*product is space*),
Lemma 127 (*product is normed vector space*),
Lemma 128 (*vector addition is continuous*).

Lemma 87 (*product is space*) is a direct dependency of:
Lemma 96 (*composition of linear maps is bilinear*),
Lemma 127 (*product is normed vector space*),
Lemma 156 (*bra-ket is bilinear map*).

Definition 88 (*inherited vector operations*) is a direct dependency of:
Lemma 90 (*space of mappings to a space*),
Lemma 93 (*space of linear maps*),
Lemma 96 (*composition of linear maps is bilinear*),
Theorem 147 (*normed vector space of continuous linear maps*),
Lemma 151 (*complete normed vector space of continuous linear maps*),
Lemma 156 (*bra-ket is bilinear map*),
Lemma 158 (*representation for bounded bilinear form*).

Lemma 90 (*space of mappings to a space*) is a direct dependency of:
Lemma 93 (*space of linear maps*).

Lemma 91 (*linear map preserves zero*) is a direct dependency of:
Lemma 99 (*kernel is subspace*),
Lemma 100 (*injective linear map has zero kernel*),
Lemma 117 (*zero on unit sphere is zero*),
Theorem 142 (*continuous linear map*),
Lemma 148 (*operator norm estimation*),
Lemma 151 (*complete normed vector space of continuous linear maps*),
Theorem 196 (*Riesz–Fréchet*).

Lemma 92 (*linear map preserves linear combinations*) is a direct dependency of:
Lemma 96 (*composition of linear maps is bilinear*),
Lemma 99 (*kernel is subspace*),
Lemma 151 (*complete normed vector space of continuous linear maps*),
Lemma 158 (*representation for bounded bilinear form*),
Lemma 186 (*orthogonal projection is continuous linear map*),
Theorem 196 (*Riesz–Fréchet*).

Lemma 93 (*space of linear maps*) is a direct dependency of:
Lemma 96 (*composition of linear maps is bilinear*).

Definition 94 (*identity map*) is a direct dependency of:
Lemma 95 (*identity map is linear map*),
Lemma 123 (*identity map is linear isometry*).

Lemma 95 (*identity map is linear map*) is a direct dependency of:
Lemma 123 (*identity map is linear isometry*).

Lemma 96 (*composition of linear maps is bilinear*) is a direct dependency of:
Lemma 150 (*compatibility of composition with continuity*).

Definition 97 (*isomorphism*) is a direct dependency of:
Theorem 196 (*Riesz–Fréchet*).

Definition 98 (*kernel*) is a direct dependency of:
Lemma 99 (*kernel is subspace*),
Lemma 100 (*injective linear map has zero kernel*),
Lemma 149 (*continuous linear maps have closed kernel*),
Theorem 196 (*Riesz–Fréchet*).

Lemma 99 (*kernel is subspace*) is a direct dependency of:
Theorem 196 (*Riesz–Fréchet*).

Lemma 100 (*injective linear map has zero kernel*) is a direct dependency of:
Theorem 196 (*Riesz–Fréchet*).

Lemma 101 (*K is space*) is a direct dependency of:
Lemma 156 (*bra-ket is bilinear map*),
Lemma 158 (*representation for bounded bilinear form*).

Definition 102 (*norm*) is a direct dependency of:
Lemma 105 (*K is normed vector space*),
Lemma 106 (*norm preserves zero*),
Lemma 107 (*norm is nonnegative*),
Lemma 108 (*normalization by nonzero*),
Lemma 111 (*norm gives distance*),
Lemma 112 (*linear span is closed*),
Lemma 118 (*reverse triangle inequality*),
Lemma 127 (*product is normed vector space*),
Lemma 128 (*vector addition is continuous*),
Lemma 129 (*scalar multiplication is continuous*),
Lemma 133 (*norm of image of unit vector*),
Lemma 134 (*norm of image of unit sphere*),
Theorem 142 (*continuous linear map*),
Theorem 147 (*normed vector space of continuous linear maps*),
Lemma 148 (*operator norm estimation*),
Lemma 151 (*complete normed vector space of continuous linear maps*),

Lemma 160 (*coercivity constant is less than continuity constant*),
Lemma 177 (*inner product gives norm*),
Theorem 180 (*orthogonal projection onto nonempty complete convex*),
Lemma 186 (*orthogonal projection is continuous linear map*),
Theorem 196 (*Riesz–Fréchet*),
Theorem 198 (*Lax–Milgram*),
Lemma 201 (*Céa*).

Definition 104 (*normed vector space*) is a direct dependency of:
Lemma 105 (*K is normed vector space*),
Lemma 106 (*norm preserves zero*),
Lemma 107 (*norm is nonnegative*),
Lemma 118 (*reverse triangle inequality*),
Lemma 127 (*product is normed vector space*),
Theorem 147 (*normed vector space of continuous linear maps*),
Lemma 158 (*representation for bounded bilinear form*),
Lemma 177 (*inner product gives norm*),
Theorem 196 (*Riesz–Fréchet*),
Theorem 198 (*Lax–Milgram*).

Lemma 105 (*K is normed vector space*) is a direct dependency of:
Lemma 154 (*topological dual is complete normed vector space*).

Lemma 106 (*norm preserves zero*) is a direct dependency of:
Lemma 134 (*norm of image of unit sphere*),
Theorem 142 (*continuous linear map*),
Lemma 148 (*operator norm estimation*),
Theorem 196 (*Riesz–Fréchet*),
Theorem 198 (*Lax–Milgram*),
Lemma 201 (*Céa*).

Lemma 107 (*norm is nonnegative*) is a direct dependency of:
Lemma 108 (*normalization by nonzero*),
Lemma 111 (*norm gives distance*),
Lemma 112 (*linear span is closed*),
Lemma 127 (*product is normed vector space*),
Lemma 133 (*norm of image of unit vector*),
Lemma 138 (*operator norm is nonnegative*),
Theorem 147 (*normed vector space of continuous linear maps*),
Lemma 148 (*operator norm estimation*),
Lemma 158 (*representation for bounded bilinear form*),
Lemma 160 (*coercivity constant is less than continuity constant*),
Theorem 180 (*orthogonal projection onto nonempty complete convex*),
Lemma 181 (*characterization of orthogonal projection onto convex*),
Lemma 186 (*orthogonal projection is continuous linear map*),
Theorem 196 (*Riesz–Fréchet*),
Theorem 198 (*Lax–Milgram*),
Lemma 201 (*Céa*).

Lemma 108 (*normalization by nonzero*) is a direct dependency of:
Lemma 117 (*zero on unit sphere is zero*),
Lemma 133 (*norm of image of unit vector*),
Theorem 196 (*Riesz–Fréchet*).

Definition 109 (*distance associated with norm*) is a direct dependency of:
Lemma 111 (*norm gives distance*),

Lemma 112 (*linear span is closed*),
Lemma 114 (*equivalent definition of closed unit ball*),
Lemma 116 (*equivalent definition of unit sphere*),
Lemma 119 (*norm is one-Lipschitz continuous*),
Lemma 128 (*vector addition is continuous*),
Lemma 129 (*scalar multiplication is continuous*),
Lemma 151 (*complete normed vector space of continuous linear maps*).

Lemma 111 (*norm gives distance*) is a direct dependency of:
Lemma 114 (*equivalent definition of closed unit ball*),
Lemma 116 (*equivalent definition of unit sphere*),
Lemma 151 (*complete normed vector space of continuous linear maps*),
Definition 194 (*Hilbert space*).

Lemma 112 (*linear span is closed*) is a direct dependency of:
Lemma 193 (*sum of complete subspace and linear span is closed*),
Lemma 202 (*finite dimensional subspace in Hilbert space is closed*).

Definition 113 (*closed unit ball*) is a direct dependency of:
Lemma 114 (*equivalent definition of closed unit ball*).

Lemma 114 (*equivalent definition of closed unit ball*) is a direct dependency of:
Theorem 142 (*continuous linear map*).

Definition 115 (*unit sphere*) is a direct dependency of:
Lemma 116 (*equivalent definition of unit sphere*).

Lemma 116 (*equivalent definition of unit sphere*) is a direct dependency of:
Lemma 117 (*zero on unit sphere is zero*),
Lemma 134 (*norm of image of unit sphere*),
Theorem 142 (*continuous linear map*),
Lemma 150 (*compatibility of composition with continuity*),
Lemma 158 (*representation for bounded bilinear form*).

Lemma 117 (*zero on unit sphere is zero*) is a direct dependency of:
Theorem 147 (*normed vector space of continuous linear maps*).

Lemma 118 (*reverse triangle inequality*) is a direct dependency of:
Lemma 119 (*norm is one-Lipschitz continuous*).

Lemma 119 (*norm is one-Lipschitz continuous*) is a direct dependency of:
Lemma 120 (*norm is uniformly continuous*).

Lemma 120 (*norm is uniformly continuous*) is a direct dependency of:
Lemma 121 (*norm is continuous*).

Lemma 121 (*norm is continuous*) is a direct dependency of:
Lemma 151 (*complete normed vector space of continuous linear maps*),
Theorem 180 (*orthogonal projection onto nonempty complete convex*).

Definition 122 (*linear isometry*) is a direct dependency of:
Lemma 123 (*identity map is linear isometry*),
Lemma 145 (*linear isometry is continuous*),
Theorem 196 (*Riesz–Fréchet*),
Theorem 198 (*Lax–Milgram*).

Lemma 123 (*identity map is linear isometry*) is a direct dependency of:
Lemma 146 (*identity map is continuous*).

Definition 124 (*product norm*) is a direct dependency of:
Lemma 127 (*product is normed vector space*),
Lemma 128 (*vector addition is continuous*).

Lemma 127 (*product is normed vector space*) is a direct dependency of:
Lemma 128 (*vector addition is continuous*).

Lemma 128 (*vector addition is continuous*) is a direct dependency of:
Lemma 151 (*complete normed vector space of continuous linear maps*).

Lemma 129 (*scalar multiplication is continuous*) is a direct dependency of:
Lemma 151 (*complete normed vector space of continuous linear maps*).

Lemma 133 (*norm of image of unit vector*) is a direct dependency of:
Lemma 134 (*norm of image of unit sphere*).

Lemma 134 (*norm of image of unit sphere*) is a direct dependency of:
Lemma 137 (*equivalent definition of operator norm*).

Definition 135 (*operator norm*) is a direct dependency of:
Lemma 137 (*equivalent definition of operator norm*),
Theorem 142 (*continuous linear map*),
Lemma 148 (*operator norm estimation*),
Lemma 151 (*complete normed vector space of continuous linear maps*),
Theorem 196 (*Riesz–Fréchet*).

Lemma 137 (*equivalent definition of operator norm*) is a direct dependency of:
Lemma 138 (*operator norm is nonnegative*),
Theorem 142 (*continuous linear map*),
Theorem 147 (*normed vector space of continuous linear maps*).

Lemma 138 (*operator norm is nonnegative*) is a direct dependency of:
Theorem 142 (*continuous linear map*).

Definition 139 (*bounded linear map*) is a direct dependency of:
Theorem 142 (*continuous linear map*),
Lemma 144 (*finite operator norm is continuous*),
Lemma 151 (*complete normed vector space of continuous linear maps*),
Lemma 158 (*representation for bounded bilinear form*),
Theorem 196 (*Riesz–Fréchet*).

Definition 140 (*linear map bounded on unit ball*) is a direct dependency of:
Theorem 142 (*continuous linear map*),
Lemma 144 (*finite operator norm is continuous*).

Definition 141 (*linear map bounded on unit sphere*) is a direct dependency of:
Theorem 142 (*continuous linear map*),
Lemma 144 (*finite operator norm is continuous*),
Theorem 147 (*normed vector space of continuous linear maps*).

Theorem 142 (*continuous linear map*) is a direct dependency of:
Lemma 144 (*finite operator norm is continuous*),
Lemma 151 (*complete normed vector space of continuous linear maps*),
Theorem 196 (*Riesz–Fréchet*).

Definition 143 (*set of continuous linear maps*) is a direct dependency of:
Lemma 144 (*finite operator norm is continuous*),
Theorem 147 (*normed vector space of continuous linear maps*).

Lemma 144 (*finite operator norm is continuous*) is a direct dependency of:
Lemma 145 (*linear isometry is continuous*),
Theorem 147 (*normed vector space of continuous linear maps*),
Lemma 150 (*compatibility of composition with continuity*),
Lemma 151 (*complete normed vector space of continuous linear maps*),
Lemma 158 (*representation for bounded bilinear form*),
Theorem 196 (*Riesz–Fréchet*).

Lemma 145 (*linear isometry is continuous*) is a direct dependency of:
Lemma 146 (*identity map is continuous*),
Theorem 196 (*Riesz–Fréchet*).

Lemma 146 (*identity map is continuous*) is a direct dependency of:
Lemma 193 (*sum of complete subspace and linear span is closed*),
Theorem 198 (*Lax–Milgram*).

Theorem 147 (*normed vector space of continuous linear maps*) is a direct dependency of:
Lemma 148 (*operator norm estimation*),
Lemma 151 (*complete normed vector space of continuous linear maps*),
Lemma 154 (*topological dual is complete normed vector space*),
Lemma 158 (*representation for bounded bilinear form*),
Lemma 193 (*sum of complete subspace and linear span is closed*),
Theorem 198 (*Lax–Milgram*).

Lemma 148 (*operator norm estimation*) is a direct dependency of:
Lemma 150 (*compatibility of composition with continuity*),
Theorem 198 (*Lax–Milgram*).

Lemma 149 (*continuous linear maps have closed kernel*) is a direct dependency of:
Theorem 196 (*Riesz–Fréchet*).

Lemma 150 (*compatibility of composition with continuity*) is a direct dependency of:
Theorem 198 (*Lax–Milgram*).

Lemma 151 (*complete normed vector space of continuous linear maps*) is a direct dependency of:
Lemma 154 (*topological dual is complete normed vector space*).

Definition 152 (*topological dual*) is a direct dependency of:
Lemma 158 (*representation for bounded bilinear form*),
Theorem 196 (*Riesz–Fréchet*).

Definition 153 (*dual norm*) is a direct dependency of:
Lemma 154 (*topological dual is complete normed vector space*),
Lemma 158 (*representation for bounded bilinear form*),
Theorem 196 (*Riesz–Fréchet*).

Lemma 154 (*topological dual is complete normed vector space*) is a direct dependency of:
Lemma 156 (*bra-ket is bilinear map*),
Lemma 158 (*representation for bounded bilinear form*),
Theorem 196 (*Riesz–Fréchet*),
Theorem 198 (*Lax–Milgram*).

Definition 155 (*bra-ket notation*) is a direct dependency of:
Lemma 156 (*bra-ket is bilinear map*),
Lemma 158 (*representation for bounded bilinear form*),
Theorem 196 (*Riesz–Fréchet*).

Lemma 156 (*bra-ket is bilinear map*) is a direct dependency of:
Lemma 158 (*representation for bounded bilinear form*),
Theorem 196 (*Riesz–Fréchet*).

Definition 157 (*bounded bilinear form*) is a direct dependency of:
Lemma 158 (*representation for bounded bilinear form*),
Lemma 160 (*coercivity constant is less than continuity constant*),
Theorem 198 (*Lax–Milgram*),
Lemma 201 (*Céa*).

Lemma 158 (*representation for bounded bilinear form*) is a direct dependency of:
Theorem 198 (*Lax–Milgram*).

Definition 159 (*coercive bilinear form*) is a direct dependency of:
Lemma 160 (*coercivity constant is less than continuity constant*),
Theorem 198 (*Lax–Milgram*),
Lemma 201 (*Céa*).

Lemma 160 (*coercivity constant is less than continuity constant*) is a direct dependency of:
Theorem 198 (*Lax–Milgram*).

Definition 161 (*inner product*) is a direct dependency of:
Lemma 166 (*inner product subspace*),
Lemma 167 (*inner product with zero is zero*),
Lemma 168 (*square expansion plus*),
Lemma 169 (*square expansion minus*),
Lemma 171 (*Cauchy–Schwarz inequality*),
Lemma 174 (*squared norm*),
Lemma 175 (*Cauchy–Schwarz inequality with norms*),
Lemma 176 (*triangle inequality*),
Lemma 177 (*inner product gives norm*),
Lemma 181 (*characterization of orthogonal projection onto convex*),
Lemma 185 (*characterization of orthogonal projection onto subspace*),
Lemma 186 (*orthogonal projection is continuous linear map*),
Lemma 188 (*trivial orthogonal complements*),
Lemma 189 (*orthogonal complement is subspace*),
Lemma 190 (*zero intersection with orthogonal complement*),
Theorem 191 (*direct sum with orthogonal complement when complete*),
Theorem 196 (*Riesz–Fréchet*),
Theorem 198 (*Lax–Milgram*),
Lemma 201 (*Céa*).

Definition 165 (*inner product space*) is a direct dependency of:
Lemma 166 (*inner product subspace*),
Theorem 180 (*orthogonal projection onto nonempty complete convex*),
Lemma 181 (*characterization of orthogonal projection onto convex*),
Theorem 196 (*Riesz–Fréchet*).

Lemma 166 (*inner product subspace*) is a direct dependency of:
Lemma 195 (*closed Hilbert subspace*).

Lemma 167 (*inner product with zero is zero*) is a direct dependency of:
Lemma 188 (*trivial orthogonal complements*),
Lemma 189 (*orthogonal complement is subspace*),
Theorem 191 (*direct sum with orthogonal complement when complete*),
Theorem 196 (*Riesz–Fréchet*).

Lemma 168 (*square expansion plus*) is a direct dependency of:
Lemma 169 (*square expansion minus*),
Lemma 170 (*parallelogram identity*),
Lemma 171 (*Cauchy–Schwarz inequality*),
Lemma 176 (*triangle inequality*),
Lemma 181 (*characterization of orthogonal projection onto convex*),
Theorem 198 (*Lax–Milgram*).

Lemma 169 (*square expansion minus*) is a direct dependency of:
Lemma 170 (*parallelogram identity*).

Lemma 170 (*parallelogram identity*) is a direct dependency of:
Theorem 180 (*orthogonal projection onto nonempty complete convex*).

Lemma 171 (*Cauchy–Schwarz inequality*) is a direct dependency of:
Lemma 175 (*Cauchy–Schwarz inequality with norms*).

Definition 172 (*square root of inner square*) is a direct dependency of:
Lemma 174 (*squared norm*),
Lemma 175 (*Cauchy–Schwarz inequality with norms*),
Lemma 177 (*inner product gives norm*),
Definition 194 (*Hilbert space*).

Lemma 174 (*squared norm*) is a direct dependency of:
Lemma 176 (*triangle inequality*),
Theorem 180 (*orthogonal projection onto nonempty complete convex*),
Lemma 181 (*characterization of orthogonal projection onto convex*),
Lemma 186 (*orthogonal projection is continuous linear map*),
Theorem 196 (*Riesz–Fréchet*).

Lemma 175 (*Cauchy–Schwarz inequality with norms*) is a direct dependency of:
Lemma 176 (*triangle inequality*),
Lemma 186 (*orthogonal projection is continuous linear map*),
Theorem 196 (*Riesz–Fréchet*).

Lemma 176 (*triangle inequality*) is a direct dependency of:
Lemma 177 (*inner product gives norm*).

Lemma 177 (*inner product gives norm*) is a direct dependency of:
Definition 194 (*Hilbert space*),
Theorem 198 (*Lax–Milgram*).

Definition 179 (*convex subset*) is a direct dependency of:
Theorem 180 (*orthogonal projection onto nonempty complete convex*),
Lemma 181 (*characterization of orthogonal projection onto convex*),
Lemma 182 (*subspace is convex*).

Theorem 180 (*orthogonal projection onto nonempty complete convex*) is a direct dependency of:
Theorem 183 (*orthogonal projection onto complete subspace*).

Lemma 181 (*characterization of orthogonal projection onto convex*) is a direct dependency of:
Lemma 185 (*characterization of orthogonal projection onto subspace*).

Lemma 182 (*subspace is convex*) is a direct dependency of:
Theorem 183 (*orthogonal projection onto complete subspace*),
Lemma 185 (*characterization of orthogonal projection onto subspace*).

Theorem 183 (*orthogonal projection onto complete subspace*) is a direct dependency of:
Definition 184 (*orthogonal projection onto complete subspace*),
Lemma 186 (*orthogonal projection is continuous linear map*),
Theorem 191 (*direct sum with orthogonal complement when complete*),
Lemma 192 (*sum is orthogonal sum*),
Lemma 193 (*sum of complete subspace and linear span is closed*),
Theorem 196 (*Riesz–Fréchet*).

Definition 184 (*orthogonal projection onto complete subspace*) is a direct dependency of:
Lemma 186 (*orthogonal projection is continuous linear map*),
Theorem 191 (*direct sum with orthogonal complement when complete*),
Theorem 196 (*Riesz–Fréchet*).

Lemma 185 (*characterization of orthogonal projection onto subspace*) is a direct dependency of:
Lemma 186 (*orthogonal projection is continuous linear map*),
Theorem 191 (*direct sum with orthogonal complement when complete*).

Lemma 186 (*orthogonal projection is continuous linear map*) is a direct dependency of:
Lemma 193 (*sum of complete subspace and linear span is closed*).

Definition 187 (*orthogonal complement*) is a direct dependency of:
Lemma 188 (*trivial orthogonal complements*),
Lemma 189 (*orthogonal complement is subspace*),
Lemma 190 (*zero intersection with orthogonal complement*),
Theorem 191 (*direct sum with orthogonal complement when complete*),
Theorem 196 (*Riesz–Fréchet*),
Theorem 198 (*Lax–Milgram*).

Lemma 188 (*trivial orthogonal complements*) is a direct dependency of:
Theorem 196 (*Riesz–Fréchet*),
Theorem 198 (*Lax–Milgram*).

Lemma 189 (*orthogonal complement is subspace*) is a direct dependency of:
Theorem 196 (*Riesz–Fréchet*).

Lemma 190 (*zero intersection with orthogonal complement*) is a direct dependency of:
Theorem 191 (*direct sum with orthogonal complement when complete*),
Lemma 193 (*sum of complete subspace and linear span is closed*).

Theorem 191 (*direct sum with orthogonal complement when complete*) is a direct dependency of:
Lemma 192 (*sum is orthogonal sum*),
Lemma 193 (*sum of complete subspace and linear span is closed*),
Theorem 196 (*Riesz–Fréchet*).

Lemma 192 (*sum is orthogonal sum*) is a direct dependency of:
Lemma 202 (*finite dimensional subspace in Hilbert space is closed*).

Lemma 193 (*sum of complete subspace and linear span is closed*) is a direct dependency of:
Lemma 202 (*finite dimensional subspace in Hilbert space is closed*).

Definition 194 (*Hilbert space*) is a direct dependency of:
Lemma 195 (*closed Hilbert subspace*),
Theorem 196 (*Riesz–Fréchet*),
Theorem 198 (*Lax–Milgram*),
Lemma 202 (*finite dimensional subspace in Hilbert space is closed*).

Lemma 195 (*closed Hilbert subspace*) is a direct dependency of:
Theorem 196 (*Riesz–Fréchet*),
Theorem 200 (*Lax–Milgram, closed subspace*).

Theorem 196 (*Riesz–Fréchet*) is a direct dependency of:
Theorem 198 (*Lax–Milgram*).

Lemma 197 (*compatible ρ for Lax–Milgram*) is a direct dependency of:
Theorem 198 (*Lax–Milgram*).

Theorem 198 (*Lax–Milgram*) is a direct dependency of:
Theorem 200 (*Lax–Milgram, closed subspace*),
Theorem 203 (*Lax–Milgram–Céa, finite dimensional subspace*).

Lemma 199 (*Galerkin orthogonality*) is a direct dependency of:
Lemma 201 (*Céa*).

Theorem 200 (*Lax–Milgram, closed subspace*) is a direct dependency of:
Lemma 201 (*Céa*),
Theorem 203 (*Lax–Milgram–Céa, finite dimensional subspace*).

Lemma 201 (*Céa*) is a direct dependency of:
Theorem 203 (*Lax–Milgram–Céa, finite dimensional subspace*).

Lemma 202 (*finite dimensional subspace in Hilbert space is closed*) is a direct dependency of:
Theorem 203 (*Lax–Milgram–Céa, finite dimensional subspace*).

Theorem 203 (*Lax–Milgram–Céa, finite dimensional subspace*)

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